

HOMOLOGICAL STABILITY OF AUTOMORPHISM GROUPS OF QUADRATIC MODULES AND MANIFOLDS

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ABSTRACT. We prove homological stability for both general linear groups of modules over a ring with finite stable rank and unitary groups of quadratic modules over a ring with finite unitary stable rank. In particular, we do not assume the modules and quadratic modules to be well-behaved in any sense: for example, the quadratic form may be singular. This extends results by van der Kallen and Mirzaii–van der Kallen respectively. Combining these results with the machinery introduced by Galatius–Randal-Williams to prove homological stability for moduli spaces of simply-connected manifolds of dimension $2n \geq 6$, we get an extension of their result to the case of virtually polycyclic fundamental groups.

1. INTRODUCTION AND STATEMENT OF RESULTS

We say that the sequence $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$ of topological spaces satisfies homological stability if the induced maps $(f_k)_*: H_k(X_n) \rightarrow H_k(X_{n+1})$ are isomorphisms for $k < An + B$ for some constants A and B . In most cases where homological stability is known it is extremely hard to compute any particular $H_k(X_n)$. However, there are several techniques to compute the stable homology groups $H_k(X_\infty)$ and homological stability can therefore be used to give many potentially new homology groups.

1.1. General Linear Groups. In [15], van der Kallen proves homological stability for the group $\mathrm{GL}_n(R)$ of R -module automorphisms of R^n . For the special case that R is a PID, Charney [4] had earlier shown homological stability. In the first part of this paper we consider the analogous homological stability problem for groups of automorphisms of general R -modules M ; we write $\mathrm{GL}(M)$ for these groups. In order to phrase our stability range we define the rank of an R -module M , $\mathrm{rk}(M)$, to be the biggest number n so that R^n is a direct summand of M . The stability range then says that the rank of M has to be big compared to the so-called stable rank of R , $\mathrm{sr}(R)$. In particular, the stable rank of R needs to be finite which holds for example for Dedekind domains and more generally algebras that are finite as a module over a commutative Noetherian ring of finite Krull dimension.

Theorem A. *The map*

$$H_k(\mathrm{GL}(M); \mathbb{Z}) \rightarrow H_k(\mathrm{GL}(M \oplus R); \mathbb{Z}),$$

induced by the inclusion $\mathrm{GL}(M) \hookrightarrow \mathrm{GL}(M \oplus R)$, is an epimorphism for $k \leq \frac{\mathrm{rk}(M) - \mathrm{sr}(R)}{2}$ and an isomorphism for $k \leq \frac{\mathrm{rk}(M) - \mathrm{sr}(R) - 1}{2}$.

For the commutator subgroup $\mathrm{GL}(M)'$ the map

$$H_k(\mathrm{GL}(M)'; \mathbb{Z}) \rightarrow H_k(\mathrm{GL}(M \oplus R)'; \mathbb{Z})$$

is an epimorphism for $k \leq \frac{\mathrm{rk}(M) - \mathrm{sr}(R) - 1}{3}$ and an isomorphism for $k \leq \frac{\mathrm{rk}(M) - \mathrm{sr}(R) - 3}{3}$.

We emphasise that M is allowed to be any module over R . For example over the integers, M could be $\mathbb{Z}/100\mathbb{Z} \oplus \mathbb{Z}^{100}$. We also get statements for polynomial and abelian coefficients. The full statement of our theorem is given in Theorem 2.9.

This part of the paper can be seen as a warm up for the heart of the algebraic part of this paper, which is homological stability for the automorphism groups of quadratic modules.

1.2. Unitary Groups. A quadratic module is a tuple (M, λ, μ) consisting of an R -module M , a sesquilinear form $\lambda: M \times M \rightarrow R$, and a function μ on M into a quotient of R , where λ measures how far μ is from being linear. The precise definition is given in Section 3.1. The basic example of a quadratic module is the *hyperbolic module* H , which is given by

$$\left(R^2 \text{ with basis } e, f; \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}; \mu \text{ determined by } \mu(e) = \mu(f) = 0 \right).$$

For a quadratic module M we write $U(M)$ for its unitary group, i.e. the group of all automorphisms that fix the quadratic structure on M . Mirzaii–van der Kallen [12] have shown homological stability for the unitary groups $U(H^n)$ and our Theorem B below extends this to general quadratic modules.

We write $g(M)$ for the Witt index of M as a quadratic module, which is defined to be the maximal number n so that H^n is a direct summand of M . In our stability range we use the notion of unitary stable rank of R , $\text{usr}(R)$, which is at least as big as the stable rank and also requires a certain transitivity condition on unimodular vectors of fixed length. Analogously to Theorem A the Witt index of M has to be big in relation to the unitary stable rank of R . In particular, $\text{usr}(R)$ needs to be finite which is the case for both examples given above of rings with finite stable rank.

Theorem B. *The map*

$$H_k(U(M); \mathbb{Z}) \rightarrow H_k(U(M \oplus H); \mathbb{Z})$$

is an epimorphism for $k \leq \frac{g(M) - \text{usr}(R) - 1}{2}$ and an isomorphism for $k < \frac{g(M) - \text{usr}(R) - 2}{2}$.

For the commutator subgroup $U(M)'$ the map

$$H_k(U(M)'; \mathbb{Z}) \rightarrow H_k(U(M \oplus H)'; \mathbb{Z})$$

is an epimorphism for $k \leq \frac{g(M) - \text{usr}(R) - 1}{2}$ and an isomorphism for $k < \frac{g(M) - \text{usr}(R) - 3}{2}$.

We again emphasise that M can be an arbitrary quadratic module – in particular, it can be singular. As in the case for general linear groups, we get an analogous statement for abelian and polynomial coefficients. The full statement is given in Theorem 3.17.

To show homological stability for both the automorphism groups of modules and quadratic modules we use the machinery developed in Randal-Williams–Wahl [14]. The actual homological stability results are straightforward applications of that paper assuming that a certain semisimplicial set is highly connected. Showing that this assumption is indeed satisfied is the main goal in Chapters 2 and 3.

1.3. Moduli Spaces of Manifolds. Our theorem in the unitary case can also be used to extend the homological stability result for moduli spaces of simply-connected manifolds of dimension $2n \geq 6$ by Galatius–Randal-Williams [7] to certain non-simply-connected manifolds.

For a compact connected smooth manifold W of dimension $2n$ we write $\text{Diff}_\partial(W)$ for the topological group of all diffeomorphisms of W that restrict to the identity near the boundary, and call its classifying space $B\text{Diff}_\partial(W)$ the *moduli space of manifolds of type W* . As in the algebraic settings described previously there is a notion of rank: Define the *genus* of W as

$$g(W) := \sup\{g \in \mathbb{N} \mid \text{there are } g \text{ disjoint embeddings of } \mathbb{S}^n \times \mathbb{S}^n \setminus \text{int}(\mathbb{D}^{2n}) \text{ into } W\}.$$

Let S denote the manifold $([0, 1] \times \partial W) \# (\mathbb{S}^n \times \mathbb{S}^n)$. We get an inclusion

$$\text{Diff}_\partial(W) \hookrightarrow \text{Diff}_\partial(W \cup_{\partial W} S)$$

by extending diffeomorphisms by the identity on S . This gluing map then has an induced map on classifying spaces which we denote by s . Galatius–Randal-Williams have shown that for simply-connected manifolds of dimension $2n \geq 6$ the induced map

$$s_*: H_k(B\text{Diff}_\partial(W)) \longrightarrow H_k(B\text{Diff}_\partial(W \cup_{\partial W} S))$$

is an epimorphism for $k \leq \frac{g(W) - 1}{2}$ and an isomorphism for $k \leq \frac{g(W) - 3}{2}$. The following extends this result to certain non-simply-connected manifolds.

Theorem C. *Let W be a compact connected manifold of dimension $2n \geq 6$. Then the map*

$$s_*: H_k(B\text{Diff}_\partial(W)) \longrightarrow H_k(B\text{Diff}_\partial(W \cup_{\partial W} S))$$

is an epimorphism for $k \leq \frac{g(W) - \text{usr}(\mathbb{Z}[\pi_1(W)])}{2}$ and an isomorphism for $k \leq \frac{g(W) - \text{usr}(\mathbb{Z}[\pi_1(W)]) - 2}{2}$.

For a virtually polycyclic fundamental group, e.g. a finitely generated abelian group, the unitary stable rank of its group ring is known to be finite by Crowley-Sixt [6]. Combining Theorem C with [7, Cor. 1.9] yields a computation of $H_k(B\text{Diff}_\partial(W))$ in the stable range.

Acknowledgements. These results will form part of my Cambridge PhD thesis. I am grateful to my supervisor Oscar Randal-Williams for many interesting and inspiring conversations and much helpful advice. I was partially supported by the Studienstiftung des deutschen Volkes and by the EPSRC.

2. HOMOLOGICAL STABILITY FOR GENERAL LINEAR GROUPS

This chapter treats the case of automorphism groups of modules. For the case of modules of the form R^n for some ring R there are several results available already, e.g. results by Charney [4] for R a Dedekind domain and by van der Kallen [15] for R with finite stable rank.

We consider the case of general modules over a ring with finite stable rank. To prove homological stability for this setting we use the machinery of Randal-Williams–Wahl [14]. This mainly involves showing the high connectivity of a certain semisimplicial set. We start by generalising a complex introduced by van der Kallen and show its high connectivity. Even though this complex is not exactly the one needed for the machinery of Randal-Williams–Wahl, it is good enough to deduce the high connectivity of that semisimplicial set. We can then immediately extract a homological stability result for various coefficients systems.

2.1. The Complex and its Connectivity. Following [15], for a set V we define $\mathcal{O}(V)$ to be the poset of ordered sequences of distinct elements in V of length at least one. The partial ordering on $\mathcal{O}(V)$ is given by refinement, i.e. we write $(w_1, \dots, w_m) \leq (v_1, \dots, v_n)$ if there is a strictly increasing map $\phi: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ such that $w_i = v_{\phi(i)}$. We say that $F \subseteq \mathcal{O}(V)$ satisfies the *chain condition* if for every element $(v_1, \dots, v_n) \in F$ and every $(w_1, \dots, w_m) \leq (v_1, \dots, v_n)$ we also have $(w_1, \dots, w_m) \in F$. For $v = (v_1, \dots, v_n) \in F$, we write F_v for the set of all sequences $(w_1, \dots, w_m) \in F$ such that $(w_1, \dots, w_m, v_1, \dots, v_n) \in F$. Note that if F satisfies the chain condition and $v, w \in F$ then $(F_v)_w = F_{vw}$. We write $F_{\leq k}$ for the subset of F containing all sequences of length $\leq k$.

We write $\text{GL}(M)$ for the group of automorphisms of general R -modules M . A sequence (v_1, \dots, v_n) of elements in M is called *unimodular* if there are R -module homomorphisms

$$f_1, \dots, f_n: R \rightarrow M \text{ and } \phi_1, \dots, \phi_n: M \rightarrow R$$

such that $f_i(1) = v_i$ and $\phi_j \circ f_i = \delta_{i,j} \cdot \mathbb{1}_R$. An element $v \in M$ is called *unimodular* if it is unimodular as a sequence in M of length 1. The condition $\phi_j \circ f_i = \delta_{i,j} \cdot \mathbb{1}_R$ holds if and only if the matrix $(\phi_j \circ f_i(1))_{i,j}$ is the identity matrix. In fact, for a sequence to be unimodular it is enough to find $\tilde{\phi}_1, \dots, \tilde{\phi}_n$ so that the matrix $(\tilde{\phi}_j \circ f_i(1))_{i,j}$ is invertible.

Lemma 2.1. *Given a sequence (v_1, \dots, v_n) in M and R -module homomorphisms*

$$f_1, \dots, f_n: R \rightarrow M \text{ and } \tilde{\phi}_1, \dots, \tilde{\phi}_n: M \rightarrow R$$

so that $f_i(1) = v_i$ and the matrix $(\tilde{\phi}_j \circ f_i(1))_{i,j}$ is invertible. Then (v_1, \dots, v_n) is already unimodular.

Proof. Let A^{-1} denote the inverse of the matrix $(\tilde{\phi}_j \circ f_i(1))_{i,j}$. We define R -module homomorphisms $\phi_j: M \rightarrow R$ as follows:

$$\phi_1 \oplus \dots \oplus \phi_n: M \xrightarrow{\tilde{\phi}_1 \oplus \dots \oplus \tilde{\phi}_n} R^n \xrightarrow{\cdot A^{-1}} R^n,$$

where $\phi_j(m)$ is the j -th entry of the vector $\phi_1 \oplus \dots \oplus \phi_n(m)$. By construction we have $\phi_j(v_i) = \delta_{i,j}$ and therefore the sequence (v_1, \dots, v_n) is unimodular. \square

Let R^∞ denote the free R -module with basis e_1, e_2, \dots and let M^∞ denote the R -module $M \oplus R^\infty$. Then we write $\mathcal{U}(M)$ for the subposet of $\mathcal{O}(M)$ consisting of unimodular sequences in M . Note that for $(v_1, \dots, v_n) \in M$ it is the same to say the sequence is unimodular in M or it is unimodular in $M \oplus R^\infty$.

Definition 2.2. A ring R satisfies the *stable range condition* (S_n) if for every unimodular vector $(r_1, \dots, r_{n+1}) \in R^{n+1}$ there are $t_1, \dots, t_n \in R$ such that the vector $(r_1 + t_1 r_{n+1}, \dots, r_n + t_n r_{n+1}) \in R^n$ is unimodular. If n is the smallest such number we say R has *stable rank* n , $\text{sr}(R) = n$ and it has $\text{sr}(R) = \infty$ if such an n does not exist.

Note that the stable range in the sense of Bass [3], (SR_n) , is the same as our stable range condition (S_{n-1}) . The absolute stable rank of a ring R , $\text{asr}(R)$, as defined by Magurn–van der Kallen–Vaserstein in [11] is an upper bound for the stable rank, i.e. $\text{sr}(R) \leq \text{asr}(R)$ ([11, Lemma 1.2]). In the following we give some of the well-known examples of rings and their stable ranks.

Examples 2.3.

- (1) A commutative Noetherian ring R of finite Krull dimension d satisfies $\text{sr}(R) \leq d+1$. In particular, if R is a Dedekind domain then $\text{sr}(R) \leq 2$ ([8, 4.1.11]) and for a field k , the polynomial ring $K = k[t_1, \dots, t_n]$ satisfies $\text{sr}(K) \leq n+1$ ([17, Thm. 8]).
- (2) More generally, any R -algebra A that is finitely generated as an R -module satisfies $\text{sr}(A) \leq d+1$, for R again a commutative Noetherian ring of finite Krull dimension d . [11, Thm. 3.1] or [8, 4.1.15]
- (3) Recall that a ring R is called *semi-local* if $R/J(R)$ is a left Artinian ring, for $J(R)$ the Jacobson radical of R . A semi-local ring satisfies $\text{sr}(R) = 1$. [8, 4.1.17]
- (4) Recall that a group G is called *virtually polycyclic* if there is a sequence of normal subgroups

$$G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_{n-1} \triangleright G_n = 0$$

such that each quotient G_i/G_{i+1} is cyclic or finite. Its *Hirsch number* $h(G)$ is the number of infinite cyclic factors. For a virtually polycyclic group G we have $\text{sr}(\mathbb{Z}[G]) \leq h(G) + 2$. [6, Thm. 7.3]

For an R -module M we define the *rank* of M as

$$\text{rk}(M) := \sup\{n \in \mathbb{N} \mid \text{there is an } R\text{-module } M' \text{ such that } M \cong R^n \oplus M'\}.$$

Using this notion we can phrase the following theorem. Here and in the following, we use the convention that the condition of a space to be n -connected for $n \leq -2$ (and so in particular for $n = -\infty$) is vacuous.

Theorem 2.4.

- (1) $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$ is $(\text{rk}(M) - \text{sr}(R) - 1)$ -connected,
- (2) $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_1, \dots, v_k)}$ is $(\text{rk}(M) - \text{sr}(R) - k - 1)$ -connected for $(v_1, \dots, v_k) \in \mathcal{U}(M^\infty)$.

In [15, Thm. 2.6 (i), (ii)] van der Kallen has proven this theorem for the special case of modules of the form R^n . Our proof of Theorem 2.4 adapts the techniques and ideas that he has used. Just as in van der Kallen's proof, we use the following technical lemma several times in the proof of Theorem 2.4.

Lemma 2.5. Let $F \subseteq \mathcal{U}(M^\infty)$ satisfy the chain condition. Let $X \subseteq M^\infty$ be a subset.

- (1) Assume that the poset $\mathcal{O}(X) \cap F$ is d -connected and that, for all sequences (v_1, \dots, v_m) in $F \setminus \mathcal{O}(X)$, the poset $\mathcal{O}(X) \cap F_{(v_1, \dots, v_m)}$ is $(d-m)$ -connected. Then F is d -connected.
- (2) Assume that for all sequences (v_1, \dots, v_m) in $F \setminus \mathcal{O}(X)$, the poset $\mathcal{O}(X) \cap F_{(v_1, \dots, v_m)}$ is $(d-m+1)$ -connected. Assume further that there is a sequence (y_0) of length 1 in F with $\mathcal{O}(X) \cap F \subseteq F_{(y_0)}$. Then F is $(d+1)$ -connected.

Proof outline. The proof of [15, Lemma 2.13] also works in this setting, where we use [15, Lemma 2.12] with the obvious modification of allowing $F \subseteq \mathcal{U}(M^\infty)$ so that it fits into our framework. \square

We are not the first ones that have the idea of showing homological stability for automorphism groups of modules more general than R^n : In [15, Rmk. 2.7 (2)] van der Kallen has suggested a possible generalisation of his results using the notion of “big” modules as defined in [18].

Proof of Theorem 2.4. Analogous to the proof of [15, Thm. 2.6] we will also show the following statements.

- (a) $\mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty)$ is $(\text{rk}(M) - \text{sr}(R))$ -connected,
- (b) $\mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty)_{(v_1, \dots, v_k)}$ is $(\text{rk}(M) - \text{sr}(R) - k)$ -connected for all (v_1, \dots, v_k) in $\mathcal{U}(M^\infty)$.

Recall that e_1 denotes the first standard basis element of R^∞ in $M^\infty = M \oplus R^\infty$.

The proof is by induction on $g = \text{rk}(M)$. Notice that statements (1), (2), (a), and (b) all hold for $g < \text{sr}(R)$ so we can assume $g \geq \text{sr}(R)$. The structure of the proof is as follows. We start by proving (b) which enables us to deduce (2). We will then proof statements (1) and (a) simultaneously by applying statement (2).

We may suppose $M = R^g \oplus M'$ for an R -module M' , since the posets in statements (1), (2), (a), and (b) only depend on the isomorphism class of M . We write x_1, \dots, x_g for the standard basis of R^g .

Proof of (b). For $Y := M \cup (M + e_1)$ we write $F := \mathcal{O}(Y) \cap \mathcal{U}(M^\infty)_{(v_1, \dots, v_k)}$. Let $d := g - \text{sr}(R) - k$, so we have to show that F is d -connected.

In the case $g = \text{sr}(R)$ we only have to consider $k = 1$. Then we have to show that F is non-empty. The strategy for this part is as follows: We define a map $f \in \text{GL}(M^\infty)$ so that Y is fixed under f as a set and the projection of $f(v_1)$ onto R^g , $f(v_1)|_{R^g}$, is unimodular. Then the sequence $(f(v_1)|_{R^g}, e_1)$ is unimodular in M^∞ . We will show that, therefore, the sequence $(f(v_1), e_1)$ is also unimodular in M^∞ and so is the sequence $(v_1, f^{-1}(e_1))$. Since $e_1 \in Y$ and the automorphism f fixes Y setwise we get $f^{-1}(e_1) \in Y$ and thus F is non-empty as it contains $f^{-1}(e_1)$.

We start by writing

$$v_1 = \sum_{i=1}^g x_i r_i + p + a,$$

where $r_i \in R$, $p \in M'$, and $a \in R^\infty$. Since v_1 is unimodular there is an R -module homomorphism $\phi: M^\infty \rightarrow R$ satisfying $\phi(v_1) = 1$. In particular,

$$1 = \phi(v_1) = \sum_{i=1}^g \phi(x_i) r_i + \phi(p + a),$$

which shows that $(r_1, \dots, r_g, \phi(p + a)) \in R^{g+1}$ is unimodular. As $g = \text{sr}(R)$ there are $t_1, \dots, t_g \in R$ such that the sequence

$$(r_1 + t_1 \phi(p + a), \dots, r_g + t_g \phi(p + a))$$

is unimodular. Now consider the map

$$\begin{aligned} M^\infty = R^g \oplus M' \oplus R^\infty &\xrightarrow{f} M^\infty = R^g \oplus M' \oplus R^\infty \\ (a_1, \dots, a_g, q, b) &\longmapsto (a_1 + t_1 \phi(q + b), \dots, a_g + t_g \phi(q + b), q, b), \end{aligned}$$

which is invertible. The map f satisfies $f(Y) = Y$ and the projection of $f(v_1)$ onto R^g is unimodular. Thus, by definition there are homomorphisms $f_1: R \rightarrow M^\infty$ and $\phi_1: M^\infty \rightarrow R$ so that $f_1(1) = f(v_1)|_{R^g}$ and $\phi_1 \circ f_1 = \mathbb{1}_R$. Note that we can assume that ϕ_1 is zero away from R^g as otherwise we can restrict to R^g before we apply ϕ_1 . This shows that the sequence $(f(v_1)|_{R^g}, e_1)$ is unimodular by choosing $\phi_2: M^\infty \rightarrow R$ to be the projection onto the coefficient of e_1 . For the sequence $(f(v_1), e_1)$ we change f_1 to map 1 to $f(v_1)$ but keeping all other homomorphisms the same then the matrix $(\phi_j \circ f_i(1))_{i,j}$ is an upper triangular matrix with 1's on the diagonal. In particular, it is invertible, so the sequence $(f(v_1), e_1)$ is unimodular by Lemma 2.1. Since f is an automorphism of M^∞ the sequence $(v_1, f^{-1}(e_1))$ is also unimodular. By construction we have $f(Y) = Y$ and so in particular $f^{-1}(e_1) \in Y$. Hence, F is non-empty as it contains $f^{-1}(e_1)$.

Now consider the case $g > \text{sr}(R)$. As in the case above there is an $f \in \text{GL}(M^\infty)$ such that $f(Y) = Y$ and $f(v_1)|_{R^g}$ is unimodular. The group $\text{GL}_g(R)$ acts transitively on the set of unimodular elements in R^g (by [16, Thm. 2.3 (c)]). This only holds in the case $g > \text{sr}(R)$ so the case $g = \text{sr}(R)$ had to be proven separately. Hence, there exists a map $\psi \in \text{GL}_g(R) \leq \text{GL}(M^\infty)$ such that $\psi(f(v_1)|_{R^g}) = x_g$. By applying $\psi \circ f$, considered as an automorphism of M^∞ , to M^∞ , without loss of generality we can assume that the projection of v_1 to R^g is x_g . We define

$$X := \{v \in Y \mid \text{the } x_g\text{-coordinate of } v \text{ vanishes}\}.$$

We now check that the assumptions of Lemma 2.5 (1) are satisfied. Notice that

$$\mathcal{U}(M^\infty)_{(v_1, \dots, v_k)} = \mathcal{U}(M^\infty)_{(v_1, v'_2, \dots, v'_k)},$$

for $v'_i = v_i + v_1 \cdot r_i$ for $r_i \in R$, as the span of v_1, v'_2, \dots, v'_k is the same as that of v_1, v_2, \dots, v_k . As the projection of v_1 to R^g is x_g , we may choose the r_i so that the x_g -coordinate of each v'_i vanishes. For $\tilde{Y} := R^{g-1} \oplus 0 \oplus M' \subset M^\infty$ we have

$$\begin{aligned} \mathcal{O}(X) \cap F &= \mathcal{O}(X) \cap \mathcal{O}(Y) \cap \mathcal{U}(M^\infty)_{(v_1, \dots, v_k)} \\ &\cong \mathcal{O}(\tilde{Y}) \cap \mathcal{U}(M^\infty)_{(v'_2, \dots, v'_k)}. \end{aligned}$$

Since we have $\text{rk}(\tilde{Y}) \geq g - 1$ we can use the induction hypothesis to see that $\mathcal{O}(\tilde{Y}) \cap \mathcal{U}(M^\infty)_{(v'_2, \dots, v'_k)}$ (and therefore $F \cap \mathcal{O}(X)$) is d -connected. Analogously, for $(w_1, \dots, w_l) \in F \setminus \mathcal{O}(X)$ we get

$$\begin{aligned} \mathcal{O}(X) \cap F_{(w_1, \dots, w_l)} &= \mathcal{O}(X) \cap \mathcal{O}(Y) \cap \mathcal{U}(M^\infty)_{(v_1, \dots, v_k, w_1, \dots, w_l)} \\ &\cong \mathcal{O}(\tilde{Y}) \cap \mathcal{U}(M^\infty)_{(v'_2, \dots, v'_k, w'_1, \dots, w'_l)}, \end{aligned}$$

which is $(d - l)$ -connected by the induction hypothesis. Therefore, Lemma 2.5 (1) shows that F is d -connected.

Proof of (2). Let us write

$$d := g - \text{sr}(R) - 1 \text{ and } X := (R^{g-1} \oplus M') \cup ((R^{g-1} + x_g) \oplus M').$$

Then we have

$$\begin{aligned} &\mathcal{O}(X) \cap (\mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_1, \dots, v_k)}) \\ &= \mathcal{O}\left((R^{g-1} \oplus M') \cup ((R^{g-1} + x_g) \oplus M')\right) \cap \mathcal{U}(M^\infty)_{(v_1, \dots, v_k)}, \end{aligned}$$

which is $(d - k)$ -connected by (b) after a change of coordinates.

Similarly, for $(w_1, \dots, w_k) \in \mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(w_1, \dots, w_l)} \setminus \mathcal{O}(X)$ we have

$$\begin{aligned} &\mathcal{O}(X) \cap (\mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_1, \dots, v_k)})_{(w_1, \dots, w_l)} \\ &= \mathcal{O}(X) \cap (\mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_1, \dots, v_k, w_1, \dots, w_l)}), \end{aligned}$$

which is $(d - k - l)$ -connected by the above. Hence, by Lemma 2.5 (1) the claim follows.

Proof of (1) and (a). By induction let us assume that statement (a) holds for $R^{g-1} \oplus M'$ and we want to it for $M = R^g \oplus M'$. Before we finish the induction for (a) we will show that this already implies statement (1) for $M = R^g \oplus M'$. For this consider d and X as in the proof of (2). Then

$$\begin{aligned} &\mathcal{O}(X) \cap (\mathcal{O}(M) \cap \mathcal{U}(M^\infty)) \\ &= \mathcal{O}\left((R^{g-1} \oplus M') \cup ((R^{g-1} + x_g) \oplus M')\right) \cap \mathcal{U}(M^\infty) \end{aligned}$$

is d -connected by (a) after a change of coordinates. The remaining assumption of Lemma 2.5 (1), i.e. that $\mathcal{O}(X) \cap (\mathcal{O}(M) \cap \mathcal{U}(M^\infty))_{(v_1, \dots, v_m)}$ is $(d - m)$ -connected, we have already shown in the proof of (2). Thus, $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$ is $(g - \text{sr}(R) - 1)$ -connected which proves statement (1).

To prove (a) we will apply Lemma 2.5 (2) for $X = M$ and $y_0 = e_1$. Consider

$$(v_1, \dots, v_k) \in \mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty) \setminus \mathcal{O}(X).$$

Without loss of generality we may suppose that $v_1 \notin X$ as otherwise we can permute the v_i . By definition of X the e_1 -coordinate of v_1 therefore equals 1. Analogous to the proof of (b) we have

$$\mathcal{O}(X) \cap \mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty)_{(v_1, \dots, v_k)} \cong \mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v'_1, \dots, v'_k)},$$

where $v'_i := v_i + v_1 r_i$ is chosen so that the e_1 -coordinate of v'_i is 0 for all i . This is $(d - k + 1)$ -connected by (1) for $k = 1$ and by (2) for $k \geq 2$. By construction we have

$$\mathcal{O}(X) \cap \mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty) \subseteq (\mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty))_{(e_1)}$$

and thus we can apply Lemma 2.5 (2) to show that $\mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty)$ is $(g - \text{sr}(R))$ -connected which proves (a).

When showing statement (a) for $M = R^g \oplus M'$ we only used statement (1) for $M = R^g \oplus M'$ which follows from (a) for $R^{g-1} \oplus M'$ so this indeed is a valid induction to show both statements (1) and (a). \square

The following propositions are consequences of the path-connectedness of $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$ and therefore, by Theorem 2.4, hold in particular for R -modules M such that $\text{rk}(M) \geq \text{sr}(R) + 1$. The statements and proofs are [7, Prop. 3.3] and [7, Prop. 3.4] respectively for the case of ordinary R -modules.

Proposition 2.6 (Transitivity). *If $\phi_0, \phi_1: R \rightarrow M$ are morphisms of R -modules and the poset $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$ is path-connected, then there is an automorphism f of M such that $\phi_1 = f \circ \phi_0$.*

Proof. Note that an R -module map $R \rightarrow M$ is defined by where it sends the unit 1 of the ring R . Suppose first that $(\phi_1(1), \phi_2(1))$ is in $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$. This implies

$$M \cong \phi_1(R) \oplus \phi_2(R) \oplus M'$$

for some R -module M' and that there is an automorphism of M which interchanges the $\phi_i(R)$ and fixes M' . Consider the equivalence relation between morphisms $f: R \rightarrow M$ of differing by an automorphism of M . We have just shown that two morphisms corresponding to two adjacent vertices in $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$ are equivalent. But the poset is path connected by assumption and hence all vertices are equivalent. \square

Proposition 2.7 (Cancellation). *Let M and N be R -modules with $M \oplus R \cong N \oplus R$. If the poset $\mathcal{O}(M \oplus R) \cap \mathcal{U}(M^\infty)$ is path-connected, then there is also an isomorphism $M \cong N$.*

Proof. As in the proof of Proposition 2.6 we can assume that the isomorphism $\phi: M \oplus R \rightarrow N \oplus R$ satisfies $\phi|_R = \text{id}_R$. Thus, by considering quotient modules we get

$$M \cong \frac{M \oplus R}{R} \cong \frac{\phi(M \oplus R)}{\phi(R)} = \frac{N \oplus R}{R} \cong N. \quad \square$$

2.2. Homological Stability. We now prove homological stability of general linear groups over modules (Theorem 2.9), which induces in particular Theorem A, using the machinery of Randal-Williams–Wahl [14]. We write $(fR\text{-Mod}, \oplus, 0)$ for the groupoid of finitely-generated right R -modules and their isomorphisms. In order to apply the main homological stability theorems in [14] we need to show that the corresponding category $UfR\text{-Mod} := \langle fR\text{-Mod}, fR\text{-Mod} \rangle$ defined in [14, Sec. 1.1] satisfies the required axioms, i.e. it is locally homogeneous and satisfies the connectivity axiom LH3. Note that local homogeneity at (M, R) for an R -module M satisfying $\text{rk}(M) \geq \text{sr}(R)$ follows from [14, Prop. 1.6] and [14, Thm. 1.8 (a), (b)]. The following lemma verifies the axiom LH3 from the connectivity of the complex considered in Theorem 2.4.

Lemma 2.8. *The semisimplicial set $W_n(M, R)_\bullet$ as defined in [14, Def. 2.1] is $\lfloor \frac{n + \text{rk}(M) - \text{sr}(R) - 2}{2} \rfloor$ -connected.*

The proof adapts the ideas of the proof of [14, Lemma 5.9]. Here, we just comment on the changes that have to be made to the proof of [14, Lemma 5.9] in order to prove the above lemma.

Proof outline. We define $X(M)_\bullet$ to be the semisimplicial set with p -simplices the split injective R -module homomorphisms $f: R^{p+1} \rightarrow M$, and with i -th face map given by precomposing with the inclusion $R^i \oplus 0 \oplus R^{p-i} \rightarrow R^{p+1}$. We write $U(M)$ for the simplicial complex with vertices the R -module homomorphisms $v: R \rightarrow M$ which are split injections (without a choice of splitting), and where a tuple (v_0, \dots, v_p) spans a p -simplex if and only if the sum $v_0 \oplus \dots \oplus v_p: R^{p+1} \rightarrow M$ is a split injection.

Note that the poset of simplices of $X(M)_\bullet$ is equal to the poset $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$ and that, given a p -simplex $\sigma = \langle v_0, \dots, v_p \rangle \in U(M)$, the poset of simplices of the complex $(\text{Link}_{U(M)}(\sigma))_\bullet^{ord}$ equals the poset $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_0, \dots, v_p)}$. Hence, by applying Theorem 2.4 and arguing as in the proof of [14, Lemma 5.9] we get that $U(M \oplus R^n)$ is weakly Cohen–Macaulay of dimension $n + \text{rk}(M) - \text{sr}(R)$.

As in the proof of [14, Lemma 5.9] we want to show that the assumptions of [9, Thm. 3.6] are satisfied. The complex $S_n(M, R)$ is a join complex over $U(M \oplus R^n)$ by the same reasoning as in the proof in [14]. In order to show that $\pi(\text{Link}_{S_n(M, R)}(\sigma))$ is weakly Cohen–Macaulay of dimension $n + \text{rk}(M) - \text{sr}(R) - p - 2$ for each p -simplex $\sigma \in S_n(M, R)$ we apply Proposition 2.7 instead of [14, Prop. 5.8] in the proof of [14, Lemma 5.9]. This shows that the remaining assumptions of [9, Thm. 3.6] are satisfied. Applying this and [14, Thm. 2.10] then yields the claim. \square

Applying Theorems [14, Thm. 3.1], [14, Thm. 3.4] and [14, Thm. 4.20] to $(UfR\text{-Mod}, \oplus, 0)$ yields the following theorem which directly implies Theorem A.

Theorem 2.9. *Let $F: UfR\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ be a coefficient system of degree r at 0 in the sense of [14, Def. 4.10]. Then for $s = \text{rk}(M) - \text{sr}(R)$ the map*

$$H_k(\text{GL}(M); F(M)) \rightarrow H_k(\text{GL}(M \oplus R); F(M \oplus R))$$

is

- (1) an epimorphism for $k \leq \frac{s}{2}$ and an isomorphism for $k \leq \frac{s-1}{2}$, if F is constant,
- (2) an epimorphism for $k \leq \frac{s-r}{2}$ and an isomorphism for $k \leq \frac{s-2-r}{2}$, if F is split polynomial in the sense of [14],
- (3) an epimorphism for $k \leq \frac{s}{2} - r$ and an isomorphism for $k \leq \frac{s-2}{2} - r$.

For the commutator subgroup $\text{GL}(M)'$ we get that the map

$$H_k(\text{GL}(M)'; F(M)) \rightarrow H_k(\text{GL}(M \oplus R)'; F(M \oplus R))$$

is

- (4) an epimorphism for $k \leq \frac{s-1}{3}$ and an isomorphism for $k \leq \frac{s-3}{3}$, if F is constant,
- (5) an epimorphism for $k \leq \frac{s-1-2r}{3}$ and an isomorphism for $k \leq \frac{s-4-2r}{3}$, if F is split polynomial in the sense of [14],
- (6) an epimorphism for $k \leq \frac{s-1}{3} - r$ and an isomorphism for $k \leq \frac{s-4}{3} - r$.

3. HOMOLOGICAL STABILITY FOR UNITARY GROUPS

The aim of this chapter is to prove the analogue of Theorem 2.9 for the case of unitary groups of quadratic modules. This again uses the machinery of Randal-Williams–Wahl introduced in [14]. In this setting we consider the complex of hyperbolic unimodular sequences in a quadratic module M . For the special case where M is a hyperbolic module this has been considered in [12]. We prove its high connectivity and deduce the assumptions for the machinery of Randal-Williams–Wahl.

3.1. The Complex and its Connectivity. Following [1] and [2] let R be a ring with an anti-involution $\bar{\cdot}: R \rightarrow R$, i.e. $\bar{\bar{r}} = r$ and $\overline{\bar{r}s} = \bar{s} \bar{r}$. Fix a unit $\varepsilon \in R$ which is a central element of R and satisfies $\bar{\varepsilon} = \varepsilon^{-1}$. Consider a subgroup Λ of $(R, +)$ satisfying

$$\Lambda_{\min} := \{r - \varepsilon \bar{r} \mid r \in R\} \subseteq \Lambda \subseteq \{r \in R \mid \varepsilon \bar{r} = -r\} =: \Lambda_{\max}$$

and $\bar{r}\Lambda r \subseteq \Lambda$. An (ε, Λ) -quadratic module is a triple (M, λ, μ) , where M is a right R -module, $\lambda: M \times M \rightarrow R$ is a sesquilinear form (i.e. λ is R -antilinear in the first variable and R -linear in the second), and $\mu: M \rightarrow R/\Lambda$ is a function, satisfying

- (1) $\lambda(x, y) = \varepsilon \overline{\lambda(y, x)}$,
- (2) $\mu(x \cdot a) = \bar{a} \mu(x) a$ for $a \in R$,
- (3) $\mu(x + y) - \mu(x) - \mu(y) = \lambda(x, y) \bmod \Lambda$.

The direct sum of two quadratic modules (M_1, λ_1, μ_1) and (M_2, λ_2, μ_2) is given by the quadratic module $(M_1 \oplus M_2, \lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2)$, where

$$\begin{aligned} (\lambda_1 \oplus \lambda_2)((m_1, m_2), (m'_1, m'_2)) &:= \lambda_1(m_1, m'_1) + \lambda_2(m_2, m'_2), \\ (\mu_1 \oplus \mu_2)(m_1, m_2) &:= \mu_1(m_1) + \mu_2(m_2), \end{aligned}$$

for $m_i, m'_i \in M_i$. The *unitary group* is defined as

$$U(M) := \{A \in \text{GL}(M) \mid \lambda(Ax, Ay) = \lambda(x, y), \mu(Ax) = \mu(x) \text{ for all } x, y \in M\}.$$

The *hyperbolic module* H over R is the (ε, Λ) -quadratic module given by

$$\left(R^2 \text{ with basis } e, f; \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}; \mu(e) = \mu(f) = 0 \right).$$

We write H^g for the direct sum of g copies of the hyperbolic module H .

Examples of unitary groups for the quadratic module H^g with various choices of $(R, \varepsilon, \Lambda)$ can be found in [12, Ex. 6.1].

Definition 3.1. A ring R satisfies the transitivity condition (T_n) if the groups $EU^\varepsilon(H^n, \Lambda)$, which is the subgroup of $U(H^n)$ consisting of elementary matrices as defined in [12, Ch. 6], acts transitively on the set

$$C_r^\varepsilon(R, \Lambda) := \{x \in R^{2n} \mid x \text{ is unimodular, } \mu(x) = r \bmod \Lambda\}$$

for every $r \in R$. The ring R has *unitary stable range* (US_n) if it satisfies the stable range condition (S_n) , as defined in Definition 2.2, as well as the transitivity condition (T_{n+1}) . We say that R has *unitary stable rank* n , $\text{usr}(R) = n$, if n is the least number such that (US_n) holds and $\text{usr}(R) = \infty$ if such an n does not exist.

The transitivity condition (T_n) and hence the unitary stable range (US_n) are conditions on $(R, \varepsilon, \Lambda)$ and not just on R . However, to make our notation consistent with the literature we write $\text{usr}(R)$ as introduced above which drops both ε and Λ .

As remarked in [12, Rmk. 6.4] we have $\text{usr}(R) \leq \text{asr}(R) + 1$ for the absolute stable rank of Magurn–van der Kallen–Vaserstein [11]. In the special case that the involution on R is the identity map (which implies that R is commutative), we have $\text{usr}(R) \leq \text{asr}(R)$. We now give some well-known examples of rings and their unitary stable rank.

Examples 3.2.

- (1) Let R be a commutative Noetherian ring of finite Krull dimension d . Then any R -algebra A that is finitely generated as an R -module satisfies $\text{usr}(A) \leq d + 2$. [11, Thm. 3.1]
- (2) A semi-local ring satisfies $\text{usr}(R) \leq 2$. [11, Thm. 2.4]
- (3) For a virtually polycyclic group G we have $\text{usr}(\mathbb{Z}[G]) \leq h(G) + 3$. [6, Thm. 7.3]

Let (M, λ, μ) be a quadratic module. A sequence (v_1, \dots, v_k) of elements in M is called *unimodular* if v_1, \dots, v_k is a basis of a free direct summand of M . We say that the sequence is λ -*unimodular* if there are elements w_1, \dots, w_k in M such that $\lambda(w_i, v_j) = \delta_{i,j}$, where $\delta_{i,j}$ denotes the Kronecker delta. We write $\mathcal{U}(M)$ for the subposet of $\mathcal{O}(M)$ consisting of unimodular sequences in M and $\mathcal{U}(M, \lambda)$ for the subposet of λ -unimodular sequences in M .

Note that every λ -unimodular sequence is in particular unimodular. The following lemma shows that there are cases where the converse is also true.

Lemma 3.3. *Let the sequence (v_1, \dots, v_k) be unimodular in M . If there is a submodule $N \subseteq M$ containing the v_i such that $\lambda|_N$ is non-singular, then the sequence (v_1, \dots, v_k) is λ -unimodular in N .*

Proof. Let (v_1, \dots, v_k) be a unimodular sequence in M . This means that there are maps $f_1, \dots, f_k: R \rightarrow M$ with $f_i(1) = v_i$ and maps $\phi_1, \dots, \phi_k: M \rightarrow R$ with $\phi_j \circ f_i = \delta_{i,j} \cdot 1_R$. Note that this implies that $\phi_j(v_i) = \delta_{i,j}$. Now, λ being non-singular on N means that the map

$$\begin{aligned} N &\longrightarrow N^* \\ v &\longmapsto \lambda(-, v) \end{aligned}$$

is an isomorphism. Hence, there are $w'_1, \dots, w'_k \in N$ such that $\lambda(-, w'_i) = \phi_i(-)$ on N . Defining $w_i := w'_i \varepsilon$ then yields

$$\lambda(w_i, v_j) = \lambda(w'_i \varepsilon, v_j) = \varepsilon \overline{\lambda(v_j, w'_i)} \varepsilon = \varepsilon \overline{\phi_i(v_j)} \varepsilon = \delta_{i,j}. \quad \square$$

We call a subset S of a quadratic module (M, λ, μ) *isotropic* if $\mu(x) = 0$ and $\lambda(x, y) = 0$ for all $x, y \in S$. Let $\mathcal{IU}(M)$ denote the set of λ -unimodular sequences (x_1, \dots, x_k) in M such that x_1, \dots, x_k span an isotropic direct summand of M . We write $\mathcal{HU}(M)$ for the set of sequences $((x_1, y_1), \dots, (x_k, y_k))$ such that $(x_1, \dots, x_k), (y_1, \dots, y_k) \in \mathcal{IU}(M)$, and $\lambda(x_i, y_j) = \delta_{i,j}$. This can also be thought of as the set of module maps $H^k \rightarrow M$. We call $\mathcal{IU}(M)$ the *poset of isotropic λ -unimodular sequences* and $\mathcal{HU}(M)$ the *poset of hyperbolic λ -unimodular sequences*.

Let $\mathcal{MU}(M)$ be the set of sequences $((x_1, y_1), \dots, (x_k, y_k)) \in \mathcal{O}(M \times M)$ satisfying

- (1) $(x_1, \dots, x_k) \in \mathcal{IU}(M)$,
- (2) for each i we have either $y_i = 0$ or $\lambda(x_j, y_i) = \delta_{j,i}$,
- (3) the span $\langle y_1, \dots, y_k \rangle$ is isotropic.

We identify the poset $\mathcal{IU}(M)$ with $\mathcal{MU}(M) \cap \mathcal{O}(M \times \{0\})$ and the poset $\mathcal{HU}(M)$ with $\mathcal{MU}(M) \cap \mathcal{O}(M \times (M \setminus \{0\}))$.

In order to phrase the main theorem of this section we introduce the following notion: For an (ε, Λ) -quadratic module (M, λ, μ) define the *Witt index* as

$$g(M) := \sup\{g \in \mathbb{N} \mid \text{there is a quadratic module } M' \text{ such that } M \cong H^g \oplus M'\}.$$

Theorem 3.4. *The poset $\mathcal{HU}(M)$ is $\lfloor \frac{g(M) - \text{usr}(R) - 3}{2} \rfloor$ -connected and for every $x \in \mathcal{HU}(M)$ the poset $\mathcal{HU}(M)_x$ is $\lfloor \frac{g(M) - \text{usr}(R) - |x| - 3}{2} \rfloor$ -connected.*

For the special case that the quadratic module M is a direct sum of hyperbolic modules H^n , Theorem 3.4 has been proven by Mirzaii-van der Kallen in [12, Thm. 7.4]. Galatius-Randal-Williams have treated the case of general quadratic modules over the integers.

In order to prove Theorem 3.4 we need the following lemma which extends [12, Lemma 6.6] to the case of general quadratic modules. Note, however, that the proof is not an extension of the proof of [12, Lemma 6.6] but rather uses techniques of Vaserstein [17]. A similar statement has been given by Petrov in [13, Prop. 6]. However, Petrov considers hyperbolic modules which are defined over rings with a pseudoinvolution and only allows $\varepsilon = -\bar{1}$. He also states his connectivity range using a different rank, called the Λ -stable rank.

Lemma 3.5. *Let $P \oplus H^g$ be a quadratic module. If $g \geq \text{usr}(R) + k$ and $(v_1, \dots, v_k) \in \mathcal{U}(P \oplus H^g, \lambda)$ then there is an automorphism $\phi \in U(P \oplus H^g)$ of the quadratic module $P \oplus H^g$ such that $\phi(v_1, \dots, v_k) \subseteq P \oplus H^k$ and the projection of $\phi(v_1, \dots, v_k)$ to the hyperbolic H^k is unimodular.*

We will prove this lemma in Section 3.2. In the remaining part of this section, we will explain how this can be used to prove Theorem 3.4. The proof of [12, Thm. 7.4] uses the connectivity of the poset of isotropic λ -unimodular sequences in the hyperbolic module H^n , $\mathcal{IU}(H^n)$, given in [12, Thm. 7.3]. We will use the following theorem, which shows the connectivity of the poset of isotropic λ -unimodular sequences in a quadratic module M .

Theorem 3.6. *The poset $\mathcal{IU}(M)$ is $\lfloor \frac{g(M) - \text{usr}(R) - 2}{2} \rfloor$ -connected and for every $x \in \mathcal{IU}(M)$ the poset $\mathcal{IU}(M)_x$ is $\lfloor \frac{g(M) - \text{usr}(R) - |x| - 2}{2} \rfloor$ -connected.*

Proof outline. The proof is analogous to the proof of [12, Thm. 7.3]. Therefore, we will only comment on the changes that need to be made in order to prove Theorem 3.6.

There are obvious modifications of [12, Lemma 6.9] and [12, Lemma 7.1] allowing quadratic modules (M, λ, μ) so that it fits into our framework. There is also analogue of [12, Lemma 7.2] in our setting which is given as follows.

Lemma 3.7. *Let $g(M) \geq \text{usr}(R) + k$. For $((x_1, y_1), \dots, (x_k, y_k)) \in \mathcal{HU}(M)$ we define $V := \langle x_1, \dots, x_k \rangle$, $W := \langle y_1, \dots, y_k \rangle$, and $P := V^\perp \cap W^\perp$. Then*

- (1) $\mathcal{IU}(M)_{(x_1, \dots, x_k)} \cong \mathcal{IU}(P)\langle V \rangle$,
- (2) $\mathcal{HU}(M) \cap \mathcal{MU}(M)_{((x_1, 0), \dots, (x_k, 0))} \cong \mathcal{HU}(M)_{((x_1, y_1), \dots, (x_k, y_k))} \langle V \times V \rangle$,
- (3) $\mathcal{HU}(M)_{((x_1, y_1), \dots, (x_k, y_k))} \cong \mathcal{HU}(P)$.

As in [12, Lemma 7.2], the proof of Lemma 3.7 follows from the obvious modifications of the proofs of [5, Lemma 3.4] and [5, Thm. 3.2]. Using those as well as Lemma 3.5 instead of [12, Lemma 6.6] the theorem follows as described in the proof of [12, Thm. 7.3]. \square

Outline of the proof of Theorem 3.4. The proof is analogous to the proof of [12, Thm. 7.4]. The only changes that need to be made are the modifications described in the proof of Theorem 3.6 as well as using Theorem 3.6 instead of [12, Thm. 7.3]. \square

3.2. Proof of Lemma 3.5. Following [17] an $(n+k) \times k$ -matrix A is called *unimodular* if it has a left inverse. Note that the matrix A is unimodular if and only if the matrix CA is unimodular for any invertible matrix $C \in \text{GL}_{n+k}(R)$. A ring R satisfies (S_n^k) if for every unimodular $(n+k) \times k$ -matrix A , there exists an element $r \in R^{n+k-1}$ such that

$$\left(\begin{array}{c|c} \mathbb{1}_{n+k-1} & r^\top \\ \hline 0 & 1 \end{array} \right) \cdot A = \begin{pmatrix} B \\ u \end{pmatrix},$$

where the matrix B is unimodular and u is the last row of A .

Note that condition (S_n^1) is the same as condition (S_n) . Furthermore, Vaserstein shows in [17, Thm. 3'] shows that the condition (S_n^k) is equivalent to the condition (S_n) .

3.2.1. $n \times k$ -Blocks. Given a quadratic R -module M we define an $n \times k$ -block for M to be an $n \times k$ -matrix $(r_{i,j})_{i,j}$ with entries in R together with k anti-linear maps $f_1, \dots, f_k: M \rightarrow R$. We will write this as

$$\begin{pmatrix} r_{1,1} & \dots & r_{1,k} \\ \vdots & & \vdots \\ r_{n,1} & \dots & r_{n,k} \\ f_1 & \dots & f_k \end{pmatrix}.$$

Note that with this notation an $n \times k$ -block has in fact $n+1$ rows, but n refers to the number of rows of the matrix. We refer to the row of maps (f_1, \dots, f_k) as the *last row* of A . Given an $(n+1) \times (n+1)$ -matrix of the form

$$\left(\begin{array}{ccc|c} s_{1,1} & \dots & s_{1,n} & m_1 \\ \vdots & & \vdots & \vdots \\ s_{n,1} & \dots & s_{n,n} & m_{2g} \\ \hline 0 & \dots & 0 & s \end{array} \right),$$

where $s, s_{i,j} \in R$, $m_i \in M$, we can act with it from the left on an $n \times k$ -block A by matrix multiplication, where we define

$$m_i \cdot f_j := f_j(m_i) \text{ and } s \cdot f_j := f_j(- \cdot \bar{s}).$$

We can act from the right on the block A with a $k \times k$ -matrix with entries in R again by matrix multiplication, where we define $f_j \cdot r$ to send an element $m \in M$ to $f_j(m) \cdot r$ for $r \in R$.

Definition 3.8. We say that an $n \times k$ -block A is *unimodular* if there is a $k \times (n+1)$ -matrix A_L of the form

$$\begin{pmatrix} r'_{1,1} & \cdots & r'_{1,n} & m'_1 \\ \vdots & & \vdots & \vdots \\ r'_{k,1} & \cdots & r'_{k,n} & m'_k \end{pmatrix}$$

with $r'_{i,j} \in R$ and $m'_i \in M$, such that $A_L \cdot A = \mathbf{1}_k$, where the multiplication is again given by matrix multiplication, with $m'_i \cdot f_j$ as defined above.

Note that the $n \times k$ -block A is unimodular if and only if one of the following blocks is unimodular:

$$\left(\begin{array}{c|c} 1 & 0 \\ 0 & \\ \vdots & \\ 0 & \\ f & \end{array} \middle| A \right), \left(\begin{array}{c|c} \mathbf{1}_n & v^\top \\ 0 & 1 \end{array} \right) \cdot A, \left(\begin{array}{c|c} C & 0 \\ 0 & 1 \end{array} \right) \cdot A, \text{ or } A \cdot \left(\begin{array}{c|c} 1 & v \\ 0 & \mathbf{1}_n \end{array} \right),$$

for a vector $v \in R^n$ and a matrix $C \in \text{GL}_n(R)$.

Definition 3.9. An $n \times k$ -block A for M has property (X) if there is a vector $m \in M^n$ such that

$$\left(\begin{array}{c|c} \mathbf{1}_n & m^\top \\ 0 & 1 \end{array} \right) \cdot A = \begin{pmatrix} B \\ u \end{pmatrix},$$

where the $n \times k$ -matrix B is unimodular and u is the last row of the block A .

Proposition 3.10. If $k + \text{sr}(R) \leq n + 1$ then for every unimodular $n \times k$ -block A has property (X).

The property (X) is preserved under certain operations as the following proposition shows (cf. proof of [17, Thm. 3']).

Proposition 3.11. Let A be an $n \times k$ -block for M . Then A has property (X) if and only if the block obtained from A by doing any of the following moves has property (X).

(1) Multiply on the left by a matrix of the form

$$\left(\begin{array}{c|c} \mathbf{1}_n & n^\top \\ 0 & 1 \end{array} \right),$$

for an element $n \in M^n$.

(2) Multiply on the left by a matrix of the form

$$\left(\begin{array}{c|c} C & 0 \\ 0 & 1 \end{array} \right),$$

for a matrix $C \in \text{GL}_n(R)$.

(3) Multiply on the right by a matrix $D \in \text{GL}_k(R)$.

Proof. Note that each of the above moves is invertible by a move of the same type. Hence, it is enough to show that if A has property (X) then so does the block obtained from A by doing one of the above moves.

Statement (1) follows from the fact that multiplying two of these matrices with last column $(n_1, 1)$ and $(n_2, 1)$ respectively yields another matrix of this form whose last column is given by $(n_1 + n_2, 1)$.

To show (2) we assume that the block A has property (X), i.e. we can write

$$A = \left(\begin{array}{c|c} \mathbf{1}_n & -m^\top \\ 0 & 1 \end{array} \right) \cdot \begin{pmatrix} B \\ u \end{pmatrix}.$$

Hence, we get

$$\left(\begin{array}{c|c} C & 0 \\ 0 & 1 \end{array} \right) \cdot A = \left(\begin{array}{c|c} C & 0 \\ 0 & 1 \end{array} \right) \cdot \left(\begin{array}{c|c} \mathbf{1}_n & -m^\top \\ 0 & 1 \end{array} \right) \cdot \left(\begin{array}{c|c} C^{-1} & 0 \\ 0 & 1 \end{array} \right) \cdot \left(\begin{array}{c|c} C & 0 \\ 0 & 1 \end{array} \right) \cdot \begin{pmatrix} B \\ u \end{pmatrix},$$

where the product of the first three matrices is

$$\left(\begin{array}{c|c} \mathbb{1}_n & -Cm^\top \\ \hline 0 & 1 \end{array} \right)$$

and the product of the last two matrices is of the form $\begin{pmatrix} B' \\ u \end{pmatrix}$, where B' is again unimodular. Thus, $(m')^\top := Cm^\top$ is the corresponding sequence for the block

$$\left(\begin{array}{c|c} C & 0 \\ \hline 0 & 1 \end{array} \right) \cdot A.$$

For (3) note that multiplying the matrix $\begin{pmatrix} B \\ u \end{pmatrix}$ on the right by D yields a matrix $\begin{pmatrix} BD \\ u' \end{pmatrix}$. If B_L is a left inverse of B then

$$D^{-1}B_LBD = \mathbb{1}_n$$

and hence BD is also unimodular. \square

Proof of Proposition 3.10. Let us write the unimodular $n \times k$ -block as

$$A = \begin{pmatrix} r_{1,1} & \cdots & r_{1,k} \\ \vdots & & \vdots \\ r_{n,1} & \cdots & r_{n,k} \\ f_1 & \cdots & f_k \end{pmatrix}.$$

The proof is by induction on k .

Let $k = 1$. Since the block A is unimodular, there is a left inverse $A_L := ((r'_1)^\top, \dots, (r'_n)^\top, (m')^\top)$ of A for vectors $r'_i \in R^k$ and $m' \in M^k$. Hence, the sequence $(r_{1,1}, \dots, r_{n,1}, f_1(m'_1)) \in R^{n+1}$ is unimodular by construction and since $n+1 > \text{sr}(R)$ there are $v_1, \dots, v_n \in R$ such that the sequence

$$(r_{1,1} + v_1 f_1(m'_1), \dots, r_{n,1} + v_n f_1(m'_1))$$

is unimodular. Defining $m_i := m' \cdot \bar{v}_i$ then yields the base case.

Let us assume that the statement is true for $k-1$ and consider the case $k > 1$. Since A is a unimodular block, in particular the first column $(r_1)^\top$ is unimodular having a left inverse $(r'_{1,1}, \dots, r'_{1,n}, m'_1)$ which is the first row of the left inverse A_L of A . Hence, the sequence $(r_{1,1}, \dots, r_{n,1}, f_1(m'_1))$ is unimodular. By assumption we have $n+1 > \text{sr}(R)$, so there is a vector $v := (v_1, \dots, v_n) \in R^n$ such that the sequence

$$r'_1 := r_{1,1} + v_1 f_1(m'_1), \dots, r_{n,1} + v_n f_1(m'_1) \in R^n$$

is unimodular. Thus, there is an $C \in \text{GL}_n(R)$ such that $Cr'_1 = (1, 0, \dots, 0)$. Consider the block

$$A_1 := \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 1 \end{array} \right) \cdot \left(\begin{array}{c|c} \mathbb{1}_n & v^\top \\ \hline 0 & 1 \end{array} \right) \cdot A.$$

Then A_1 is of the form

$$\left(\begin{array}{c|c} 1 & u' \\ \hline 0 & \\ \vdots & A' \\ 0 & \\ f_1 & \end{array} \right)$$

for an $(n-1) \times (k-1)$ -block A' for M . Now, by Proposition 3.11 the block A satisfies property (X) if and only if the block A_1 does. Proposition 3.11 also implies that this is equivalent to the block

$$A_2 := A_1 \cdot \left(\frac{1}{0} \middle| \frac{-u'}{\mathbf{1}_n} \right) = \left(\frac{1}{0} \middle| \frac{0}{A''} \right)$$

satisfying property (X). Therefore, it is enough to show that the block A_2 satisfies property (X). Since the block A is unimodular, so is A_2 as remarked above. This implies that the block A'' is unimodular as well. Hence, by the induction hypothesis there is a vector $m \in M^{n-1}$ such that

$$\left(\frac{1}{0} \middle| \frac{-u'}{\mathbf{1}_n} \right) \cdot A'' = \begin{pmatrix} \tilde{B} \\ \tilde{u} \end{pmatrix},$$

where the matrix \tilde{B} is unimodular and \tilde{u} is the last row of A'' . Thus,

$$\left(\frac{1}{0} \middle| \frac{0}{\mathbf{1}_{n-1}} \middle| \frac{0}{m^\top} \right) \cdot A_2 = \begin{pmatrix} 1 & 0 \\ * & \tilde{B} \\ * & \tilde{u} \end{pmatrix},$$

where the matrix $\begin{pmatrix} 1 & 0 \\ * & \tilde{B} \end{pmatrix}$ is unimodular since \tilde{B} is unimodular. □

The next proposition is an extension of [17, Thm. 1].

Proposition 3.12. *Let $k + \text{sr}(R) = n + 1$ and $l > 0$ then for any unimodular $(n + l) \times k$ -block A there is a vector $m \in M^n$ such that*

$$\left(\frac{\mathbf{1}_n}{0} \middle| \frac{0}{\mathbf{1}_l} \middle| \frac{m^\top}{1} \right) \cdot A = \begin{pmatrix} B \\ u \end{pmatrix},$$

where the $(n + l) \times k$ -matrix B is unimodular and u is the last row of the block A .

Proof. Since A is a unimodular $(n + l) \times k$ -block by Proposition 3.10 there is an element $\tilde{m} \in M^{n+l}$ such that

$$\left(\frac{\mathbf{1}_{n+l}}{0} \middle| \frac{\tilde{m}^\top}{1} \right) \cdot A = \begin{pmatrix} B_1 \\ u_1 \end{pmatrix},$$

where the $(n + l) \times k$ -matrix B_1 is unimodular and $u_1 = u$ is the last row of the block A . Since $l > 0$ and $n + l - k \geq \text{sr}(R)$ we can now apply the condition (S_{n+l-k}^k) to the unimodular matrix B_1 to get an element $v \in R^{n+l-1}$ such that

$$\left(\frac{\mathbf{1}_{n+l-1}}{0} \middle| \frac{v^\top}{1} \right) \cdot B_1 = \begin{pmatrix} B_2 \\ u_2 \end{pmatrix},$$

where the $(n + l - 1) \times k$ -matrix B_2 is unimodular and u_2 is the last row of the matrix B_1 . Together we get

$$\left(\frac{\mathbf{1}_{n+l-1}}{0} \middle| \frac{v^\top}{1} \middle| \frac{0}{1} \right) \cdot \left(\frac{\mathbf{1}_{n+l}}{0} \middle| \frac{\tilde{m}^\top}{1} \right) \cdot A = \begin{pmatrix} B_2 \\ u_2 \\ u_1 \end{pmatrix}.$$

Notice that the product of the first two matrices can be written in the form

$$\left(\begin{array}{c|c|c} \mathbb{1}_{n+l-1} & * & * \\ \hline 0 & 1 & * \\ \hline 0 & 0 & 1 \end{array} \right),$$

where the last column has entries in the module M and the rest of the matrix has entries in the ring R . Iterating this yields a matrix

$$C := \left(\begin{array}{c|cc|c} \mathbb{1}_n & & * & * \\ \hline & 1 & * & * \\ 0 & 0 & \ddots & * \\ & 0 & 0 & 1 \\ \hline 0 & & 0 & 1 \end{array} \right)$$

and $C \cdot A$ is a matrix of the form $\begin{pmatrix} B' \\ B'' \end{pmatrix}$, where B' is an $n \times k$ -matrix and B'' is an $(l+1) \times k$ -block. The matrix B' is unimodular by construction. Note that row operations involving only the rows of B'' do not change the matrix B' . Hence, we can change the above matrix C to be of the form

$$C' := \left(\begin{array}{c|c|c} \mathbb{1}_n & * & * \\ \hline 0 & \mathbb{1}_l & 0 \\ \hline 0 & 0 & 1 \end{array} \right).$$

Again, $C' \cdot A$ is a matrix of the form $\begin{pmatrix} B' \\ \tilde{B}'' \end{pmatrix}$, where B' is the same matrix as above and hence unimodular. Instead of dividing this matrix into the first n and the last $l+1$ rows, let us now divide it into the first $n+l$ and the last row, written as $\begin{pmatrix} B''' \\ u \end{pmatrix}$, where u is by construction the last row of the matrix A . Since the matrix B' is unimodular, so is the matrix B''' . Row operations on B''' correspond to multiplying B''' on the left by invertible matrices which keeps the matrix unimodular. Hence, we can perform row operations on C' using all but the last row to get a matrix of the form

$$\left(\begin{array}{c|c|c} \mathbb{1}_n & 0 & m^\top \\ \hline 0 & \mathbb{1}_l & 0 \\ \hline 0 & 0 & 1 \end{array} \right).$$

This finishes the proof. □

3.2.2. Orthogonal Transvections. Following [11, Ch. 7] let e and u be elements in the quadratic module (M, λ, μ) satisfying $\mu(e) = 0$ and $\lambda(e, u) = 0$. For $x \in \mu(u)$ we define an automorphism $\tau(e, u, x): M \rightarrow M$ of the quadratic module M by

$$\tau(e, u, x)(v) = v + u\lambda(e, v) - e\bar{\varepsilon}\lambda(u, v) - e\bar{\varepsilon}x\lambda(e, v).$$

If e is λ -unimodular, the map $\tau(e, u, x)$ is called an *orthogonal transvection*.

The following is the last ingredient in order to prove Lemma 3.5.

Proposition 3.13. ([14, Prop. 5.12]) *Let M be a quadratic module and $M \oplus H \cong H^{g+1}$. If $g \geq \text{usr}(R)$ then $M \cong H^g$.*

Proof of Lemma 3.5. We adapt the ideas of Step 1 in the proof of [11, Thm. 8.1]. Let us write

$$v_i = p_i + \sum_{l=1}^g e_l A_l^i + \sum_{l=1}^g f_l B_l^i \quad \text{for } p_i \in P \text{ and } A_l^i, B_l^i \in R.$$

As the sequence (v_1, \dots, v_k) is λ -unimodular, there are

$$w_i = q_i + \sum_{l=1}^g e_l a_l^i + \sum_{l=1}^g f_l b_l^i \quad \text{for } q_i \in P \text{ and } a_l^i, b_l^i \in R$$

satisfying

$$\begin{aligned} \delta_{i,j} &= \lambda(w_i, v_j) = (q_i, a_1^i, b_1^i, \dots, b_g^i) \begin{pmatrix} \lambda|_P & 0 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & \varepsilon & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_j \\ A_1^j \\ B_1^j \\ \vdots \\ B_g^j \end{pmatrix} \\ &= \lambda(q_i, p_j) + \sum_{l=1}^g a_l^i B_l^j + \varepsilon \sum_{l=1}^g b_l^i A_l^j \end{aligned}$$

Hence, the $(2g) \times k$ -block for P

$$\begin{pmatrix} A_1^1 & \cdots & A_1^k \\ B_1^1 & \cdots & B_1^k \\ \vdots & & \vdots \\ B_g^1 & \cdots & B_g^k \\ \lambda(-, p_1) & \cdots & \lambda(-, p_k) \end{pmatrix}$$

is unimodular. Since $g - k + 1 > \text{sr}(R)$ by Proposition 3.12 there are $\tilde{p}_1, \dots, \tilde{p}_g \in P$ such that

$$\begin{pmatrix} & \tilde{p}_1 \\ & 0 \\ \mathbb{1}_{2g} & \vdots \\ & \tilde{p}_g \\ & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1^1 & \cdots & A_1^k \\ B_1^1 & \cdots & B_1^k \\ \vdots & & \vdots \\ B_g^1 & \cdots & B_g^k \\ \lambda(-, p_1) & \cdots & \lambda(-, p_k) \end{pmatrix} = \begin{pmatrix} B \\ u \end{pmatrix},$$

where the matrix B is unimodular. Now, for $y_i \in \mu(\tilde{p}_i)$ consider the following composition of transvections

$$\tilde{\phi} := \tau(e_g, -\tilde{\varepsilon}\tilde{p}_g, \varepsilon y_g \tilde{\varepsilon}) \circ \dots \circ \tau(e_1, -\tilde{\varepsilon}\tilde{p}_1, \varepsilon y_1 \tilde{\varepsilon}).$$

Then by induction we have

$$\tilde{\phi}(v_i) = v_i + \sum_{j=1}^g \left(-\tilde{\varepsilon}\tilde{p}_j B_j^i - e_j \lambda(\tilde{p}_j, p_i) - \left(e_j \tilde{\varepsilon} \sum_{l=1}^{j-1} \lambda(\tilde{p}_j, \tilde{p}_l) B_l^i \right) - e_j y_j \tilde{\varepsilon} B_j^i \right),$$

where we have used the identity $\tilde{\varepsilon}\varepsilon = 1$ several times. Thus, $\tilde{\phi}(v_i)$ is obtained from v_i by adding multiples of the \tilde{p}_j 's to the element p_i in the non-hyperbolic summand P of M and adding $e_j \lambda(\tilde{p}_j, p_i)$'s as well as a multiple (which is independent of i) of the B_j^i 's for all $1 \leq j \leq g$. Since the multiples of the B_j^i 's are independent of i this corresponds to row operations of the matrix B and hence does not change the property of being unimodular. Hence, the projection of $\tilde{\phi}(v_1, \dots, v_k)$ to H^g , which we denote by $(\tilde{v}_1, \dots, \tilde{v}_k)$, is unimodular by definition of the \tilde{p}_i 's. Now apply [12, Lemma 6.6] to get a hyperbolic basis $\{\tilde{e}_1, \tilde{f}_1, \dots, \tilde{e}_g, \tilde{f}_g\}$ of H^g such that $\tilde{v}_1, \dots, \tilde{v}_k \in \langle \tilde{e}_1, \tilde{f}_1, \dots, \tilde{e}_g, \tilde{f}_g \rangle =: U$. Note that this does not need to be the standard basis of H^g hence we need to find an automorphism ψ of H^g that sends the above basis $\tilde{e}_1, \tilde{f}_1, \dots, \tilde{e}_g, \tilde{f}_g$ to the standard basis in H^g . Then $\phi := (\mathbb{1}_P \oplus \psi) \circ \tilde{\phi}$ will be the required automorphism.

Let V denote an orthogonal complement of U in H^g , i.e. $U \oplus V \cong H^g$. We have $g - k \geq \text{usr}(R)$ and hence Proposition 3.13 implies $V \cong H^{g-k}$. Let ψ denote the automorphism of H^g which sends U to the first k copies of H in H^g and V to the last $g - k$ copies. Using the above definition of ϕ we then have $\phi(v_1, \dots, v_k) \subseteq P \oplus H^k$ and the projection of $\phi(v_1, \dots, v_k)$ to H^k is unimodular. \square

We get the following version of [11, Thm. 8.1], but phrased in terms of the unitary stable rank instead of the absolute stable rank.

Corollary 3.14. *Let (M, λ, μ) be a quadratic module satisfying $g(M) \geq \text{usr}(R) + 1$. Then $U(M)$ acts transitively on the set of all λ -unimodular elements in M of a given length.*

Proof. For $g = g(M)$ there is a quadratic module P such that $M := P \oplus H^g$. We write $e_1, f_1, \dots, e_g, f_g$ for the basis of H^g . We show that we can map a λ -unimodular element v to $e_1 + bf_1$ with b so that it is of the same length as v .

By Lemma 3.5 there is an automorphism $\phi \in U(P \oplus H^g)$ such that $\phi(v) \subseteq P \oplus H$ and the projection of $\phi(v)$ to the hyperbolic H is unimodular. Hence, by the transitivity condition (T_g) we can map the projection of $\phi(v)$ (considered in H^g) to $e_1 + b'f_1$ having the same length as the projection of $\phi(v)$. Thus, we have mapped v to the element of same length $p + e_1 + b'f_1$ for some element $p \in P$. Applying the orthogonal transvection $\tau(f_1, -p, x)$ for some $x \in \mu(p)$ to v we get $e_1 + bf_1$, with $b = b' + \bar{\varepsilon}\lambda(p, p) - \bar{\varepsilon}x$. By construction, this is of the same length as v . \square

By Lemma 3.3 this is a generalisation of [7, Prop. 3.3] which treats the special case of quadratic modules over the integers. Note that our bound is slightly better than the bound given in the special case.

Adapting the proof of [11, Cor. 8.3], using Corollary 3.14 instead of [11, Thm. 8.1] yields the following improvement to Proposition 3.13. Note that Step 6 of [11, Thm. 8.1] still works in our setting.

Corollary 3.15. *Let M and N be quadratic modules and $M \oplus H \cong N \oplus H$. If $g(M) \geq \text{usr}(R)$ then $M \cong N$.*

In contrast to Proposition 3.13, both M and N can be general quadratic modules and, in particular, both can be non-hyperbolic modules. As in the previous corollary, this bound is slightly better than the bound given in [7, Prop. 3.4] which only treats the case $R = \mathbb{Z}$.

3.3. Homological Stability. We now show homological stability for unitary groups over quadratic modules (Theorem 3.17). This induces in particular Theorem B. As in the previous chapter we use the machinery of Randal-Williams–Wahl [14]. Let $(R, \varepsilon, \Lambda)$ -Quad be the groupoid of quadratic modules over $(R, \varepsilon, \Lambda)$ and their isomorphisms. We write $f(R, \varepsilon, \Lambda)$ -Quad for the full subcategory on those quadratic modules which are finitely generated as R -modules. Since this is a braided monoidal category it has an associated pre-braided category $Uf(R, \varepsilon, \Lambda)$ -Quad.

By Corollary 3.15 and [14, Thm. 1.8 (a)] the category $Uf(R, \varepsilon, \Lambda)$ -Quad is locally homogeneous at (M, H) for $g(M) \geq \text{usr}(R) + 1$. Axiom LH3 is verified by the following Lemma which for the special case of hyperbolic modules is shown in [14, Lemma 5.13].

Lemma 3.16. *Let M be a quadratic module with $g(M) \geq \text{usr}(R) + 1$. Then the semisimplicial set $W_n(M, H)_\bullet$ is $\lfloor \frac{n+g(M)-\text{usr}(R)-3}{2} \rfloor$ -connected.*

Proof. As in the proof of [14, Lemma 5.13], the poset of simplices of the semisimplicial set $W_n(M, H)_\bullet$ is equal to the poset $\mathcal{HU}(M \oplus H^n)$ considered in Section 3.1. Hence, they have homeomorphic geometric realisations. The claim now follows from Theorem 3.4. \square

Applying Theorems [14, Thm. 3.1], [14, Thm. 3.4] and [14, Thm. 4.20] to the quadratic module $(Uf(R, \varepsilon, \Lambda)\text{-Quad}, \oplus, 0)$ yields the following theorem which directly implies Theorem B.

Theorem 3.17. *Let M be a quadratic module satisfying $g(M) \geq \text{usr}(R) + 1$. For a coefficient system $F: Uf(R, \varepsilon, \Lambda)\text{-Quad} \rightarrow \mathbb{Z}\text{-Mod}$ of degree r at 0 in the sense of [14, Def. 4.10]. Then for $s = g(M) - \text{usr}(R)$ the map*

$$H_k(U(M); F(M)) \rightarrow H_k(U(M \oplus H); F(M \oplus H))$$

is

- (1) an epimorphism for $k \leq \frac{s-1}{2}$ and an isomorphism for $k \leq \frac{s-2}{2}$, if F is constant,
- (2) an epimorphism for $k \leq \frac{s-r-1}{2}$ and an isomorphism for $k \leq \frac{s-r-3}{2}$, if F is split polynomial in the sense of [14],
- (3) an epimorphism for $k \leq \frac{s-1}{2} - r$ and an isomorphism for $k \leq \frac{s-3}{2} - r$.

For the commutator subgroup $U(M)'$ we get that the map

$$H_k(U(M)'; F(M)) \rightarrow H_k(U(M \oplus R)'; F(M \oplus R))$$

is

- (4) an epimorphism for $k \leq \frac{s-1}{3}$ and an isomorphism for $k \leq \frac{s-3}{3}$, if F is constant,
- (5) an epimorphism for $k \leq \frac{s-2r-1}{3}$ and an isomorphism for $k \leq \frac{s-2r-4}{3}$, if F is split polynomial in the sense of [14],
- (6) an epimorphism for $k \leq \frac{s-1}{3} - r$ and an isomorphism for $k \leq \frac{s-4}{3} - r$.

4. HOMOLOGICAL STABILITY FOR MODULI SPACES OF HIGH DIMENSIONAL MANIFOLDS

Let P be a closed $(2n - 1)$ -dimensional manifold, and let W and M be compact connected $2n$ -dimensional manifolds with identified boundaries $\partial W = P = \partial M$. Following Galatius–Randal-Williams [7] we say that M and W are *stably diffeomorphic relative to P* if there is a diffeomorphism

$$W \# W_g \cong M \# W_h$$

relative to P , for some $g, h \geq 0$, where $W_g := \#_g(\mathbb{S}^n \times \mathbb{S}^n)$ for $g \geq 0$. Let \mathcal{M} denote the set of $2n$ -dimensional submanifolds $M \subset (-\infty, 0] \times \mathbb{R}^\infty$ such that

- (1) $M \cap (\{0\} \times \mathbb{R}^\infty) = \{0\} \times P$ and M contains $(-\varepsilon, 0] \times P$ for some $\varepsilon > 0$,
- (2) the boundary of M is precisely $\{0\} \times P$, and
- (3) M is stably diffeomorphic to W relative to P .

We use the topology on \mathcal{M} as described in [7, Ch. 6]. We write $\mathcal{M}(X)$ for the model of the classifying space $B\text{Diff}_\partial(X)$ defined in [7]. With this notion \mathcal{M} is the union of all $\mathcal{M}(X)$ for compact connected manifolds X with boundary P which are stably diffeomorphic to W relative to P . The stabilisation map is the same as considered in [7] and is given as follows: We choose a submanifold $S \subset [-1, 0] \times \mathbb{R}^\infty$ with collared boundary $\partial S = \{-1, 0\} \times P = S \cap (\{-1, 0\} \times \mathbb{R}^\infty)$, such that S is diffeomorphic relative to its boundary to $([-1, 0] \times P) \# W_1$. If P is not path connected, we also choose in which path component to perform the connected sum. Gluing S then induces the self-map

$$\begin{aligned} s = - \cup S: \mathcal{M} &\longrightarrow \mathcal{M} \\ M &\longmapsto (M - e_1) \cup S, \end{aligned}$$

that is, translation by one unit in the first coordinate direction followed by union of submanifolds of $(-\infty, 0] \times \mathbb{R}^\infty$. Note that by construction we have $M \cup_P S \cong M \# W$ relative to P , and hence $M \cup_P S$ is stably diffeomorphic to W if and only if M is.

As in the previous chapters, we have a notion of genus: Writing $W_{g,1} := W_g \setminus \text{int}(\mathbb{D}^{2n})$ the *genus* of a compact connected $2n$ -dimensional manifold W is

$$g(W) := \sup\{g \in \mathbb{N} \mid \text{there are } g \text{ disjoint copies of } W_{1,1} \text{ in } W\}$$

and the *stable genus* of W is

$$\bar{g}(W) := \sup_{k \geq 0} \{g(W \# W_k) - k\}.$$

Note that since $\bar{g}(W)$ is bounded above by $\frac{b_n(W)}{2}$, where $b_n(W)$ is the n -th Betti number of W , and the map $k \mapsto g(W \# W_{1,1}) - k$ is non-decreasing, the above supremum is obtained. The following theorem shows homological stability for the spaces $\mathcal{M}_g \subset \mathcal{M}$, which are the manifolds in \mathcal{M} of stable genus precisely g . Note that by definition of the stable genus, the map s defined above restricts to a map $s: \mathcal{M}_g \rightarrow \mathcal{M}_{g+1}$. For the case of simply-connected compact manifolds Galatius–Randal-Williams have shown homological stability for the spaces \mathcal{M}_g in [7, Thm. 6.2]. Our extension of their theorem is as follows.

Theorem 4.1. *Let $2n \geq 6$ and W be a compact connected manifold. Then the map*

$$s_*: H_k(\mathcal{M}_g) \longrightarrow H_k(\mathcal{M}_{g+1})$$

is an epimorphism for $k \leq \frac{g - \text{usr}(\mathbb{Z}[\pi_1(W)])}{2}$ and an isomorphism for $k \leq \frac{g - \text{usr}(\mathbb{Z}[\pi_1(W)]) - 2}{2}$.

As in [7], this implies that for any manifold W with boundary P , the restriction

$$s: \mathcal{M}(W) \longrightarrow \mathcal{M}(W \cup_P S)$$

induces an epimorphism on homology for degrees $k \leq \frac{\bar{g}(W) - \text{usr}(\mathbb{Z}[\pi_1(W)])}{2}$ and an isomorphism for degrees $k \leq \frac{\bar{g}(W) - \text{usr}(\mathbb{Z}[\pi_1(W)]) - 2}{2}$. Since $g(W) \leq \bar{g}(W)$ this implies Theorem C.

Using Example 3.2 (3) we get a special case of Theorem 4.1.

Corollary 4.2. *Let $2n \geq 6$ and W be a compact connected manifold whose fundamental group is virtually polycyclic of Hirsch length h . Then the map*

$$s_*: H_k(\mathcal{M}_g) \longrightarrow H_k(\mathcal{M}_{g+1})$$

is an epimorphism for $k \leq \frac{g-h-3}{2}$ and an isomorphism for $k \leq \frac{g-h-5}{2}$.

This theorem applies in particular to all compact connected manifolds with finite fundamental group and more generally with finitely generated abelian fundamental group.

Another consequence of the above theorem is the following cancellation result which in the case of simply-connected manifolds has been done in [7, Cor. 6.3]. The statement is closely related to [6, Thm. 1.1].

Corollary 4.3. *Let $2n \geq 6$ and P be a $(2n-1)$ -dimensional manifold. Let W and W' be compact connected manifolds with boundary P such that $W \# W_g \cong W' \# W_g$ relative to P for some $g \geq 0$. If $\bar{g}(W) \geq \text{usr}(\mathbb{Z}[\pi_1(W)]) + 2$, then $W \cong W'$ relative to P .*

Proof. Analogous to the proof of [7, Cor. 6.3], where we apply Theorem 4.1 instead of [7, Thm. 6.2]. \square

The proof of Theorem 4.1 will follow that of [7, Thm. 6.2] which treats the case of simply-connected manifolds. The idea is to consider the group of immersions of $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \text{int}(\mathbb{D}^{2n})$ into a manifold. Equipping this with a bilinear form that counts intersections and a function that counts self-intersections we get a quadratic module. The precise construction is the content of the following section. The high connectivity shown in the previous chapter then implies a connectivity statement for a complex of geometric data associated to the manifold. This is the crucial result in order to show homological stability which we do in Section 4.2.

4.1. Associating a Quadratic Module to a Manifold. In order to relate the objects in this chapter to the algebraic objects considered in Section 3.1 we want to associate to each compact connected $2n$ -dimensional manifold W a quadratic module $(\mathcal{I}_n^f(W), \lambda, \mu)$ with form parameter $((-1)^n, \Lambda_{\min})$. This will be a $\mathbb{Z}[\pi_1(W, *)]$ -module given by a version of the group of immersed framed n -spheres in W , with pairing given by the intersection form, and quadratic form given by counting self-intersections, both considered over the group ring $\mathbb{Z}[\pi_1(W, *)]$. For the rest of this chapter we drop the basepoint $*$ from the notation and just write $\pi_1(W)$.

To make this construction precise we fix a framing $b_{\mathbb{S}^n \times \mathbb{D}^n} \in \text{Fr}(\mathbb{S}^n \times \mathbb{D}^n)$ at the basepoint in $\mathbb{S}^n \times \mathbb{D}^n$ as defined in [7, Ch. 5]. We can now generalise [7, Def. 5.2], following the construction in the proof of [19, Thm. 5.2].

Definition 4.4. Let $2n \geq 6$ and W be a compact connected $2n$ -dimensional manifold, equipped with a *framed basepoint*, i.e. a point $b_W \in \text{Fr}(W)$, and an orientation compatible with b_W .

- (1) We consider the ring $\mathbb{Z}[\pi_1(W)]$ with involution given by $\bar{g} := w_1(g)g^{-1} \in \mathbb{Z}[\pi_1(W)]$, where $w_1(g)$ is the first Stiefel–Whitney class of g . Recall that the first Stiefel–Whitney class can be viewed as the homomorphism $\pi_1(W) \rightarrow \mathbb{Z}^\times = \{-1, 1\}$ which sends a loop to 1 if and only if it is orientation preserving.

Let $\mathcal{I}_n^{fr}(W)$ be the set of regular homotopy classes of immersions $i: \mathbb{S}^n \times \mathbb{D}^n \looparrowright W$ equipped with a tether in $\text{Fr}(W)$ from $Di(b_{\mathbb{S}^n \times \mathbb{D}^n})$ to b_W . The (abelian) group structure on $\mathcal{I}_n^{fr}(W)$ is given by forming the connected sum (of the cores $\mathbb{S}^n \times \{0\}$) along the framed tether given by the path to b_W .

The $\pi_1(W)$ -action is given as follows. For an orientation-preserving loop γ , lift γ to a loop based at b_W in $\text{Fr}(W)$ and concatenate this with the tether of i . For an orientation-reversing loop do the same after changing the map i by a reflection on the \mathbb{D}^n factor.

- (2) Let $a, b \in \mathcal{I}_n^{fr}(W)$ be two framed immersed spheres, whose cores we may suppose meet in general position, i.e. transversely in a finite set of points. We write c_a and c_b for the core of a and b respectively. For a point p in c_a let $\gamma_a(p)$ denote a path from the basepoint $*$ to p in c_a . Since $2n \geq 6$ such a path is canonical up to homotopy. For $a, b \in \mathcal{I}_n^{fr}(W)$ and $p \in c_a \cap c_b$ we define $\gamma_{(a,b)}(p)$ to be the concatenation of $\gamma_a(p)$ followed by the inverse of $\gamma_b(p)$.

Let us fix an orientation of M at the basepoint $*$ and transport the orientation to p along c_a . Then $\varepsilon_{(a,b)}(p)$ is defined to be the sign of the intersection of c_a and c_b with respect to this orientation at p . Given these notions we define a map

$$\begin{aligned} \lambda: \mathcal{I}_n^{fr}(W) \times \mathcal{I}_n^{fr}(W) &\longrightarrow \mathbb{Z}[\pi_1(W)] \\ (a, b) &\longmapsto \sum_{p \in c_a \cap c_b} \varepsilon_{(a,b)}(p) \gamma_{(a,b)}(p). \end{aligned}$$

- (3) Let $a \in \mathcal{I}_n^{fr}(W)$ be an immersed sphere in general position and let $p \in \mathbb{S}^n \times \{0\}$ be a point in the core of $\mathbb{S}^n \times \mathbb{D}^n$. We write $\gamma(p)$ for the path from the basepoint $*$ to p in the universal cover of the image of a in W .

At a self-intersection point of the core of a two branches of a cross. By choosing an order of these branches we can define $\varepsilon(p, q)$ as above. Recall that $\Lambda_{\min} = \{a - \varepsilon \bar{a} \mid a \in \mathbb{Z}[\pi_1(W)]\}$. We define a map

$$\begin{aligned} \mu: \mathcal{I}_n^{fr}(W) &\longrightarrow \mathbb{Z}[\pi_1(W)] / \Lambda_{\min} \\ a &\longmapsto \sum_{\substack{\{p,q\} \subset \mathbb{S}^n \\ a(p)=a(q) \\ p \neq q}} \varepsilon(p, q) \gamma(p, q), \end{aligned}$$

where $\gamma(p, q)$ is the loop in c_a based at the basepoint $*$ given by the concatenation of $a(\gamma(p))$ and the inverse of $a(\gamma(q))$, see Figure 1. The definition of Λ_{\min} guarantees that the order of the points p, q is not relevant, i.e. we have $\varepsilon(p, q) \gamma(p, q) \equiv \varepsilon(q, p) \gamma(q, p) \pmod{\Lambda_{\min}}$.

Remarks 4.5.

- (1) The proof of [19, Thm. 5.2 (i)] shows that both maps λ and μ are well-defined.
- (2) We show that we can always change a by an isotopy so that every point in $c_a \cap c_b$ yields a summand in $\lambda(a, b)$, i.e. so that no two intersection points give summands that cancel. The idea is to pair up intersection points that give the same element in $\mathbb{Z}[\pi_1(W)]$ but with opposite signs, and to use the Whitney trick to kill these intersection points. Figure 2 shows a sector of c_a and c_b in W (for c_a that is a sector of $a(\mathbb{S}^n \times \{0\})$) and the path of a to the basepoint $*$ and analogously for c_b) with two intersection points p and q . Both paths $\gamma_{(a,b)}(p)$ and $\gamma_{(a,b)}(q)$ correspond to g in $\pi_1(W)$ and the points p and q have opposite signs. If this is the case the loop e is contractible. Hence,

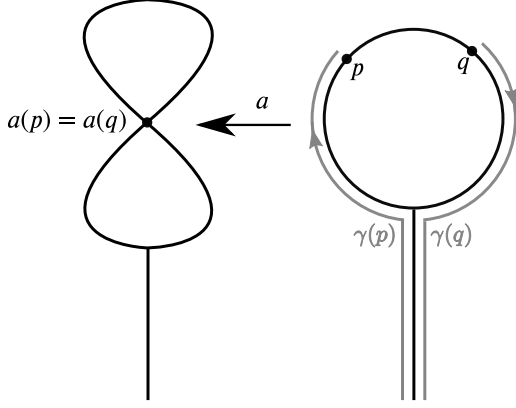
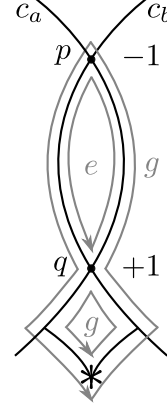
FIGURE 1. Definition of $\gamma(p, q)$.

FIGURE 2. Using the Whitney trick.

we can fill in a 2-disc and use the Whitney trick in order to move c_a away from c_b in the sector shown in the picture.

The subsequent lemma generalises [7, Lemma 5.3]. The proof is analogous to the proof of [7, Lemma 5.3], again using [19, Thm. 5.2].

Lemma 4.6. *The triple $(\mathcal{I}_n^{fr}(W), \lambda, \mu)$ is a $((-1)^n, \Lambda_{\min})$ -quadratic module.*

4.2. Proof of Theorem 4.1. As described in [7, Ch. 5] we get a map

$$K^\delta(W) \longrightarrow \mathcal{HU}(\mathcal{I}_n^{fr}(W)),$$

where $K^\delta(W)$ denotes the simplicial complex as defined in [7, Def. 5.1] and $\mathcal{HU}(\mathcal{I}_n^{fr}(W))$ is the simplicial complex defined in Section 3.1. We use this map to deduce the connectivity of $|K_\bullet^\delta(W)| = |K^\delta(W)|$ from the connectivity of $\mathcal{HU}(\mathcal{I}_n^{fr}(W))$ which we have shown in Section 3.1. This is the content of the next theorem. For the case of simply-connected manifolds this has been done in [7, Lemma 5.5], [7, Thm. 5.6], and [7, Cor. 5.9].

Theorem 4.7. *Let $2n \geq 6$ and W be a compact connected $2n$ -dimensional manifold. Then the following spaces, defined in [7, Ch. 5], are all $\lfloor \frac{\bar{g}(W) - \text{usr}(\mathbb{Z}[\pi_1(W)]) - 3}{2} \rfloor$ -connected:*

- (1) $|K_\bullet^\delta(W)|$,
- (2) $|K_\bullet(W)|$,
- (3) $|\bar{K}_\bullet(W)|$.

For the proof of this theorem we want a modified version of Theorem 3.4 using the following definition: The *stable Witt index* of a quadratic module M is

$$\bar{g}(M) := \sup_{k \geq 0} \{g(M \oplus H^k) - k\}.$$

By definition we have $g(M) \leq \bar{g}(M)$ and if the stable Witt index is big enough we in fact have equality, as the following corollary shows.

Lemma 4.8. *If $\bar{g}(M) \geq \text{usr}(R)$ then we have $g(M) \geq \bar{g}(M)$.*

Proof. For $g = \bar{g}(M)$ we know that $M \oplus H^k \cong P \oplus H^g \oplus H^k$ for some k . If $k = 0$ we immediately get $g(M) \geq g$. If $k > 0$ we get $M \oplus H^{k-1} \cong P \oplus H^g \oplus H^{k-1}$ by Corollary 3.15. Applying this argument inductively then yields $g(M) \geq g$. \square

Using the above correspondence between the Witt index and the stable Witt index we can now state Theorem 3.4 in terms of the stable Witt index.

Corollary 4.9. *The poset $\mathcal{HU}(M)$ is $\lfloor \frac{\bar{g}(M) - \text{usr}(R) - 3}{2} \rfloor$ -connected and $\mathcal{HU}(M)_x$ is $\lfloor \frac{\bar{g}(M) - \text{usr}(R) - |x| - 3}{2} \rfloor$ -connected for every $x \in \mathcal{HU}(M)$.*

Remark 4.10. Analogous to the above we can define the *stable rank* of an R -module M as

$$\overline{\text{rk}}(M) := \sup_{k \geq 0} \{\text{rk}(M \oplus R^k) - k\}.$$

As in the case of the stable Witt index this coincides with the rank of M if $\overline{\text{rk}}(M) \geq \text{sr}(R)$. This can be shown similarly to the proof of Lemma 4.8 by inductively applying Theorem 2.4 and Proposition 2.7. Using this we get a version of Theorem 2.4 in terms of the stable rank.

Proof of Theorem 4.7. For $|K_\bullet^\delta(W)|$ the proof is an extension of the proof of [7, Lemma 5.5] and hence we just comment on the changes we have to make in order to show the above statement.

Note that the complex $K^a(\mathcal{I}_n^{fr}(W), \lambda, \mu)$ as defined in [7, Def. 3.1] is the same as $\mathcal{HU}(\mathcal{I}_n^{fr}(W))$. In the following we write K^a for this complex.

For $\bar{g} = \bar{g}(\mathcal{I}_n^{fr}(W), \lambda, \mu)$ we have $\bar{g}(W) \leq \bar{g}$ and hence it is sufficient to show that $|K_\bullet^\delta(W)|$ is $\lfloor \frac{\bar{g} - \text{usr}(\mathbb{Z}[\pi_1(W)]) - 3}{2} \rfloor$ -connected.

For $k \leq \frac{\bar{g} - \text{usr}(\mathbb{Z}[\pi_1(W)]) - 3}{2}$ we consider a map $f: \partial I^{k+1} \rightarrow |K_\bullet^\delta(W)|$. By using Corollary 4.9 instead of [7, Thm. 3.2] we get a nullhomotopy $\bar{f}: I^{k+1} \rightarrow |K^a|$ with the same properties as in the proof in [7] which we have to show lifts to a nullhomotopy $F: I^{k+1} \rightarrow |K_\bullet^\delta(W)|$ of f . Note that by the second statement of Corollary 4.9 we have

$$\text{ICM}(K^a) \geq \left\lfloor \frac{\bar{g} - \text{usr}(\mathbb{Z}[\pi_1(W)])}{2} \right\rfloor \geq k + 1.$$

The construction of this lift is analogous to the proof of [7, Lemma 5.5] so we just need to explain how to extend the arguments that involve the simple connectivity of W . One of these arguments is the application of the Whitney trick. We can still apply the Whitney trick in our case, but we have to use it over the group ring $\mathbb{Z}[\pi_1(W)]$ as described in Remark 4.5 (2). The other argument is the extension from the embedding $W_{1,1} \hookrightarrow W$ to the embedding $H \hookrightarrow W$, for H as defined in [7, Ch. 5]. For this, note that the image of $\bar{f}(v_i) = h$ in K^a comes with two paths to the basepoint, one from $h(e)$ and one from $h(f)$. The proof in [7] forgets both path and chooses a new one later on (which works since W is simply-connected and hence oriented). Instead, we can keep track of the path coming from $h(e)$. This can be viewed as an embedding $[-1, 0] \times \{0\} \hookrightarrow W$. This then has a thickening by definition which gives an embedding $H \hookrightarrow W$. As in the proof of [7, Lemma 5.5] we can now conclude the connectivity range.

The proof for the case $|K_\bullet^\delta(W)|$ is an easy extension of the proof of [7, Thm. 5.6], where we use Corollary 4.9 instead of [7, Thm. 3.2] and hence get a slightly weaker connectivity range.

The remaining case follows exactly as in [7, Cor. 5.9]. \square

Outline of the proof of Theorem 4.1. The proof of [7, Thm. 6.2] only uses the assumption of W being simply-connected in [7, Lemma 6.8]. Since the proof of Theorem 4.1 is an extension of the proof of [7, Thm. 6.2], we only need to show that the map given in [7, Lemma 6.8] is $\lfloor \frac{\bar{g} - \text{usr}(\mathbb{Z}[\pi_1(W)]) - 1}{2} \rfloor$ -connected for a general compact connected manifold W of dimension $2n \geq 6$. This follows analogously to the proof of [7, Lemma 6.8] using Theorem 4.7 (3) instead of [7, Cor. 5.9]. \square

Remark 4.11. We can combine the above results with the results from Kupers in [10] for homeomorphisms, PL-homeomorphisms and homeomorphisms as a discrete group of high-dimensional manifolds. Note that the machinery in Kupers' paper does not rely on the manifolds being simply-connected but rather the input does (i.e. the connectivity of a certain complex uses simply-connectivity). Therefore, by using our more general theorem (Theorem 3.4) as the input, we can replace the assumption of the manifold being simply-connected by the group ring of the fundamental group having finite unitary stable rank.

4.3. Tangential Structures and Abelian Coefficient Systems. In the remaining part of this chapter we extend Theorem 4.1 in two different ways. One is by considering moduli spaces of manifolds with some additional structure and the other is by taking homology with coefficients in certain local coefficient systems. We follow [7, Ch. 7] and extend several of their definitions and results to our setting.

Using the notion of tangential structures we can define the space $\mathcal{M}^\theta(\hat{\ell}_P)$ of compact connected manifolds $M \in \mathcal{M}$ equipped with a θ -structure extending $\hat{\ell}_P$ for a fixed pair $(P, \hat{\ell}_P)$.

Let $\hat{\ell}_S$ be a θ -structure on the cobordism S which is standard when pulled back along the canonical embedding $\phi': W_{1,1} \rightarrow S$, in the sense of [7, Def. 7.2]. Writing $\hat{\ell}_P$ for its restriction to $\{0\} \times P \subset S$, and $\hat{\ell}'_P$ for its restriction to $\{1\} \times P$, we obtain the following map

$$(4.1) \quad \begin{aligned} s = - \cup (S, \hat{\ell}_S): \mathcal{M}^\theta(\hat{\ell}'_P) &\longrightarrow \mathcal{M}^\theta(\hat{\ell}_P) \\ (M, \hat{\ell}_M) &\longmapsto ((M - e_1) \cup S, \hat{\ell}_M \cup \hat{\ell}_S). \end{aligned}$$

As in [7] we define the θ -genus for compact connected manifolds with θ -structure as

$$g^\theta(M, \hat{\ell}_M) = \max \left\{ g \in \mathbb{N} \mid \begin{array}{l} \text{there are } g \text{ disjoint copies of } W_{1,1} \text{ in } M, \\ \text{each with standard } \theta\text{-structure} \end{array} \right\}$$

and the *stable* θ -genus as

$$\bar{g}^\theta(M, \hat{\ell}_M) = \max \{ g^\theta((M, \hat{\ell}_M) \natural_k (W_{1,1}, \hat{\ell}_{W_{1,1}})) - k \mid k \in \mathbb{N} \},$$

where the boundary connected sum is formed with k copies of $W_{1,1}$ each equipped with a standard θ -structure $\hat{\ell}_{W_{1,1}}$. As in the previous chapter, we can use the function \bar{g}^θ to consider $\mathcal{M}^\theta(\hat{\ell}_P)$ and $\mathcal{M}^\theta(\hat{\ell}'_P)$ as graded spaces in the sense of [7, Def. 6.5]

Since the spaces considered here are usually disconnected and do not have a preferred basepoint, local coefficients can be considered as a functor from the fundamental groupoid to the category of abelian groups. Note that this is closely related to the corresponding definitions in [14]. Then an *abelian coefficient system* is a local coefficient system which has trivial monodromy along all commutators, i.e. it has trivial monodromy along all nullhomologous loops.

Given a local coefficient system \mathcal{L} on $\mathcal{M}^\theta(\hat{\ell}_P)$, we can consider twisted homology with coefficients on \mathcal{L} , and the map s defined in (4.1) induces a map

$$(4.2) \quad s_*: H_k(\mathcal{M}^\theta(\hat{\ell}'_P)_g; s^*\mathcal{L}) \longrightarrow H_k(\mathcal{M}^\theta(\hat{\ell}_P)_{g+1}; \mathcal{L}).$$

For the case of simply-connected compact manifolds Galatius–Randal-Williams have shown in [7, Thm. 7.5] that the above map is an isomorphism in a range. Our extension of their theorem is as follows.

Theorem 4.12. *Let $2n \geq 6$ and W be a compact connected manifold.*

- (1) *If \mathcal{L} is abelian then the stabilisation map (4.2) is an epimorphism for $k \leq \frac{g - \text{usr}(\mathbb{Z}[\pi_1(W)])}{3}$ and an isomorphism for $k \leq \frac{g - \text{usr}(\mathbb{Z}[\pi_1(W)]) - 3}{3}$.*
- (2) *If θ is spherical in the sense of [7, Def. 7.4] and \mathcal{L} is constant, then the stabilisation map (4.2) is an epimorphism for $k \leq \frac{g - \text{usr}(\mathbb{Z}[\pi_1(W)])}{2}$ and an isomorphism for $k \leq \frac{g - \text{usr}(\mathbb{Z}[\pi_1(W))] - 2}{2}$.*

As in [7] we write $\mathcal{M}^\theta(W; \hat{\ell}'_P)$ for the space of all pairs $(M, \hat{\ell})$ with $M \in \mathcal{M}(W)$ and a bundle map $\hat{\ell}$ that restricts to $\hat{\ell}'_P$ over the boundary. Given an $\hat{\ell}_W$ so that $(W, \hat{\ell}_W)$ is such a pair, we write $\mathcal{M}^\theta(W, \hat{\ell}) \subset \mathcal{M}^\theta(W; \hat{\ell}'_P)$ for the path component containing $(W, \hat{\ell}_W)$. Then the map (4.1) restricts to a map

$$s: \mathcal{M}^\theta(W, \hat{\ell}_W) \longrightarrow \mathcal{M}^\theta(W \cup_P S, \hat{\ell}_W \cup \hat{\ell}_S)$$

and by Theorem 4.12 this is an isomorphism on homology with (abelian) coefficients in a range of degrees depending on $\bar{g}^\theta(W, \hat{\ell}_W)$.

The following proposition extends [7, Prop. 7.15] and is the analogue of Theorem 4.7 (3).

Proposition 4.13. *Let $2n \geq 6$ and let M be a compact connected $2n$ -dimensional manifold. Then the space $|\bar{K}_\bullet(M, \hat{\ell}_M)|$ as defined in [7, Def. 7.14] is $\lfloor \frac{\bar{g}(M, \hat{\ell}_M) - \text{usr}(\mathbb{Z}[\pi_1(M)]) - 3}{2} \rfloor$ -connected.*

Outline of the proof. Analogous to the proof of [7, Prop. 7.15]. In this case we need to show the analogue of Theorem 4.7 (1), that $|K^\delta(M, \hat{\ell}_M)|$, defined in [7, Def. 7.14] is $\lfloor \frac{\bar{g}(M, \hat{\ell}_M) - \text{usr}(\mathbb{Z}[\pi_1(M)] - 3)}{2} \rfloor$ -connected. All we need to change is quoting the corresponding statement from Section 3.1 and Chapter 4. In detail, this is Theorem 4.7 (1) instead of [7, Thm. 5.6], Theorem 4.7 (2) instead of [7, Cor. 5.9], Definition 4.4 instead of [7, Def. 5.2], and Theorem 3.4 instead of [7, Thm. 3.2]. \square

Outline of the proof of Theorem 4.12. This proof is based on the proof of [7, Thm. 7.5] and we therefore just describe the changes that we have to make to that proof. Note that the simply-connected assumption is only used in [7, analogue of Lemma 6.8] so we only have to show that the map considered in that statement is $\lfloor \frac{\bar{g}^\theta(W, \hat{\ell}_W) - \text{usr}(\mathbb{Z}[\pi_1(W)] - 1)}{2} \rfloor$ -connected for a compact and connected manifold W of dimension $2n \geq 6$. But this follows analogously to the proof of [7, analogue of Lemma 6.8] by applying Proposition 4.13 instead of [7, Prop. 7.15] as in the original proof. This also explains the slightly lower bound in our case. Note that throughout this proof we need to replace [7, Thm. 6.2] in the proof of [7, Thm. 7.5] by Theorem 4.1. \square

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